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***Lower bounding problems for stress constrained
discrete structural topology optimization problems***

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Lower bounding problems for stress constrained discrete structural topology optimization problems

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Abstract

The multiple load structural topology design problem is modeled as a minimization of the weight of the structure subject to equilibrium constraints and restrictions on the local stresses and nodal displacements. The problem involves a large number of discrete design variables and is modeled as a non-convex mixed 0–1 program. For this problem, several convex and mildly non-convex continuous relaxations are presented. Reformulations of these relaxations, obtained by using duality results from semi-definite and second order cone programming, are also presented. The reformulated problems are suitable for implementation in a nonlinear branch and bound framework for solving the considered class of problems to global optimality.

Key words: Topology optimization, Stress constraints, Relaxations, Global optimization

1 Introduction

In this paper we consider structural topology optimization problems in which the design variables are chosen from a finite set of given values. In particular, our interest is in minimum weight problems with constraints on the global stiffness of the structure, i.e., the compliance, and on local stress properties, such as the von Mises stress. From an optimization modeling point of view, we consider these problems to be mixed 0–1 non-convex programs.

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The main intention of this paper is to present continuous relaxations of the considered discrete problem. The second objective is to present reformulations of these relaxations, i.e., lower bounding problems, which produce good lower bounds and which are suitable for implementation within a branch and cut method capable of computing guaranteed global optimal solutions to medium- or even large-scale problem instances.

The considered class of problems suffers, besides the discrete design variables, from two additional complications. Due to the equilibrium equations, the problems are intrinsically non-convex and the natural continuous relaxations do not satisfy standard constraint qualifications such as the Mangasarian-Fromovitz or linear independence constraint qualifications [6]. The second complication is that the stress constraints must vanish if the corresponding element in the structure does not contain any material (and re-appear if material is indeed present) otherwise the feasible set becomes too stringent, see e.g. [18].

There are, at least, two possible approaches to obtain continuous relaxations of the considered discrete problem. In the first approach, the original non-convex problem is first equivalently reformulated, via additional continuous variables and linear inequality constraints, into a convex mixed discrete problem, as outlined in [13], [16], and [17]. From this equivalent formulation a convex continuous relaxation is immediate. This leads to convex quadratically constrained quadratic programs that are significantly larger, both in terms of variables and constraints, than the original formulation. In these relaxations the equilibrium equations and the stress constraints are both weakened (unless all the design variables are either zero or one). Preliminary computational results in [16] and [17] indicate that only relatively small-scale instances can be solved by branch and bound methods based exclusively on these relaxations. The main reason being the weakness of the relaxations caused by relaxing the equilibrium equations. However, the relaxations can be strengthened by adding valid inequalities, based on mechanical reasoning, and specialized combinatorial Bender's cuts as in [15].

In the second approach, the original problem is relaxed by first removing the complicating stress and displacement constraints and then relaxing the integer requirements. The resulting relaxation is a continuous minimum weight problem with constraints on the compliances. This is still a non-convex problem which, because of the equilibrium equations, in general, does not satisfy standard constraint qualifications. The two complicating issues can be resolved by reformulating the relaxation. The closely related problem of minimizing the compliance subject to a volume constraint has been extensively studied in the case of continuous variables. For this class of problems there exist several convex (and also non-convex) reformulations. Minimum compliance problems can be reformulated as semi-definite programs [8, 9, 10], second order cone programs [7], quadratically constrained quadratic programs [3] and [14], and non-smooth programs [1]. Similar refor-

mulation techniques are used in this paper to produce several reformulations of the proposed continuous minimum weight problem. In these relaxations the equilibrium equations and the compliance constraints are always satisfied but the stress constraints are completely ignored. The computational results reported in [4] and [5] for the classical minimum compliance problems indicate that relatively large-scale instances can be solved to global optimality by a branch and bound method and that this class of relaxations is fairly strong.

There is, in general, no precise relation between these two different classes of relaxations in terms of the strength. Which one has the largest lower bound (and hence is likely to lead to faster convergence of the branch and bound method) depends on the problem data and the previous branching decisions. The branch and cut method will be based on combining the different relaxations in the sense that all available relaxations are solved at each node in the search tree. The most promising solution, e.g. the solution with the largest lower bound, is then used. This, of course, creates additional over-head in the implementation and more time is spent at each node compared to a classical nonlinear branch and cut method. However, we believe that this additional effort is justified for the considered class of difficult problems.

1.1 Notation

The space of real $d \times d$ matrixes is denoted by \mathcal{M}^d . The subspace of symmetric matrices in \mathcal{M}^d is denoted by \mathcal{S}^d . We use \mathcal{S}_+^d to denote the set of symmetric positive semi-definite matrices. For $A, B \in \mathcal{S}^d$ we write $A \succeq B$ if $A - B \in \mathcal{S}_+^d$. The trace of a matrix A is denoted by $\text{Tr}(A)$. The inner product on \mathcal{M}^d is given by

$$A \bullet B = \sum_{i,j} A_{ij} B_{ij} = \text{Tr}(A^T B).$$

By e , we denote a column vector of all ones whose dimension is clear from the context of its use. The j -th unit vector is denoted by e_j . We use $(C)^+$ (and $(C)^-$) to denote the componentwise max (min) of zero and the entries in the matrix C . Throughout, we will use minimize instead of inf and maximize instead of sup to emphasize that we are interested in both the optimal solution and the optimal value.

2 Problem statement

We state the optimization problem in a format based on a finite element discretization of the continuum problem. In the ground structure approach we start with a set of n finite elements in two or three-dimensional design space with appropriate support conditions, see e.g. [12]. For a given vector $x \in \mathbb{R}^n$ of design variables with $x_1, \dots, x_n \geq 0$ the stiffness matrix of the structure in global coordinates is denoted by $K(x) \in \mathcal{S}_+^d$. Here, d denotes the number of degrees of freedom of the structure after the deletion of fixed degrees of freedom. We throughout assume that the stiffness matrix $K(x)$ depends linearly on x , i.e.,

$$K(x) := K_0 + \sum_{j=1}^n x_j K_j \in \mathcal{S}_+^d \quad (\text{A1})$$

where

$$x_j K_j \in \mathcal{S}_+^d \quad (\text{A2})$$

is the symmetric and positive semi-definite stiffness matrix of the j -th element and $K_0 \in \mathcal{S}_+^d$ is given. The design variables are interpreted as

$$x_j = \begin{cases} 1 & \text{if the } j\text{-th element contains material, and} \\ 0 & \text{otherwise.} \end{cases}$$

We consider M static load cases where the loads are given by the vectors

$$f_1, \dots, f_M \in \mathbb{R}^d \setminus \{0\} \quad (\text{A3})$$

in global reduced coordinates. The elastic equilibrium equations for the structure subjected to the static external load vector f_k is assumed to be given by

$$K(x)u_k = f_k \quad k = 1, \dots, M,$$

where $u_k \in \mathbb{R}^d$ denotes the nodal displacement vector corresponding to f_k .

We consider the following minimum weight problem where the design variables x are chosen from the set of n -dimensional binary vectors \mathbb{B}^n .

$$\begin{aligned}
 & \text{minimize} && \sum_{j=1}^n x_j \rho_j \\
 & \quad \quad \quad x \in \mathbb{B}^n, \\
 & \quad \quad \quad u_1, \dots, u_M \in \mathbb{R}^d \\
 (\text{P}) \quad & \text{subject to} && K(x)u_k = f_k \quad \forall k \\
 & && f_k^T u_k \leq \bar{\gamma}_k \quad \forall k \\
 & && u_k^T W_j u_k \leq \bar{\sigma}^2 \quad \forall k, \forall j : x_j = 1 \\
 & && \underline{u} \leq u_k \leq \bar{u} \quad \forall k \\
 & && Ax \leq b \\
 & && x \in \{0, 1\}^n.
 \end{aligned}$$

The density ρ_j for the j -th element is assumed to be strictly positive, i.e.,

$$\rho_j > 0 \quad \forall j = 1, \dots, n. \quad (\text{A4})$$

The given bounds on the nodal displacements $\underline{u} \in \mathbb{R}^d$ and $\bar{u} \in \mathbb{R}^d$ are assumed to satisfy

$$-\infty < \underline{u}_i \leq \bar{u}_i < +\infty \quad \forall i = 1, \dots, d. \quad (\text{A5})$$

Here, $A \in \mathbb{R}^{m \times n}$ is a given matrix and $b \in \mathbb{R}^m$ is a given vector. The pair (A, b) can be used to include valid inequalities and cuts in the problem formulation in a branch and cut framework. The compliances $f_k^T u_k$ for the different load cases are bounded by the given scalars

$$\bar{\gamma}_k > 0 \quad \forall k = 1, \dots, M. \quad (\text{A6})$$

The stress constraints are described by the given symmetric and positive semi-definite matrices $W_j \in \mathcal{S}_+^d$ and the stress bound

$$\bar{\sigma} > 0. \quad (\text{A7})$$

Notice that the stress constraints are only included in the formulation if the corresponding design variable is equal to one, i.e., if the corresponding element contains material.

We make no assumptions on the feasibility of (P). If the upper bounds on the stresses and/or compliances are too tight, or if the ground structure is not sufficiently "rich", the feasible set of (P) may very well be empty.

The above formulation should be considered as a simplified model intended to ease the notational burden in the following presentation. Several generalizations of the formulation, which are important from a mechanical modeling point of view, are indeed possible without significantly changing the main results. For example, more than one stress constraint per element could be included. It is also possible to include other (linear) constraints involving the displacements, for example strain constraints. Finally, the model can be extended to include more than one binary variable per element. This is useful if, for each element, several material choices are available.

3 Continuous relaxation and reformulations

The first class of relaxations of the discrete problem (P) is obtained by first removing some of the complicating constraints from the formulation, in this case the stress constraints and the displacement bounds, and then relaxing the constraints $x \in \{0, 1\}^n$ to $x \in [0, 1]^n$. The resulting continuous relaxation is a non-convex minimum weight problem with compliance constraints. Several convex (and non-convex) reformulations of this relaxation, suitable for a practical implementation of a nonlinear branch and bound method, are then presented.

3.1 Statement of the first relaxation

A natural continuous non-convex relaxation of the discrete minimum weight problem (P) is given by the program

$$\begin{aligned}
 & \text{minimize} && \rho^T x \\
 & \quad \quad \quad x \in \mathbb{R}^n, \\
 & \quad \quad \quad u_1, \dots, u_M \in \mathbb{R}^d \\
 \text{(R)} \quad & \text{subject to} && K(x)u_k = f_k \quad \forall k \\
 & && f_k^T u_k \leq \bar{\gamma}_k \quad \forall k \\
 & && Ax \leq b \\
 & && x_j \in [\underline{x}_j, \bar{x}_j] \quad \forall j.
 \end{aligned}$$

Here, $\underline{x}_j \in \mathbb{R}$ and $\bar{x}_j \in \mathbb{R}$ are lower and upper bounds on the design variables. For these bounds we assume that

$$0 \leq \underline{x}_j \leq \bar{x}_j < +\infty \quad \forall j = 1, \dots, n. \quad (\text{A8})$$

These bounds are included for generality and can be used to fix some of the variables to zero or one in a branch and cut framework. The j -th variable is fixed to zero (one) if $\underline{x}_j = \bar{x}_j = 0$ ($\underline{x}_j = \bar{x}_j = 1$). If $\underline{x}_j = 0$ and $\bar{x}_j = 1$ for all j then we obtain the "root" relaxation of the mixed integer program (P).

The continuous relaxation (R) is intrinsically non-convex due to the bilinear forms in the equilibrium equations. The relaxation (R) does not, in general, satisfy standard constraint qualifications since the stiffness matrix may become positive semi-definite. Note that the feasible set of the continuous relaxation (R) may be empty.

3.2 Reformulations of the first relaxation

The first reformulation of the relaxation (R) is based on eliminating the state variables u_1, \dots, u_M and by rewriting the compliance and equilibrium constraints by minimum potential energy. Let $c_k(x)$ denote the compliance for the k -th load condition as a function of x only. Then, $c_k(x)$ is given by

$$c_k(x) = \sup_{u \in \mathbb{R}^d} \{2f_k^T u - u^T K(x)u\}.$$

For a formal proof of this statement, see [2]. The second step in the reformulation is to write the compliance constraint as a matrix inequality. A more general version and a proof of the following Lemma are found in [9].

Lemma 1. *Let K be a symmetric positive semi-definite $d \times d$ matrix, let $f \in \mathbb{R}^d$, and let*

$$c = \sup_{u \in \mathbb{R}^d} \{2f^T u - u^T K u\}.$$

Then the inequality $c \leq \bar{\gamma}$ is equivalent to positive semi-definiteness of the matrix

$$\begin{pmatrix} \bar{\gamma} & f^T \\ f & K \end{pmatrix}.$$

The continuous relaxation (R) can thus equivalently be rewritten as the linear semi-definite program

$$\begin{aligned}
& \underset{x \in \mathbb{R}^n}{\text{minimize}} && \rho^T x \\
\text{(R-SDP)} \quad & \text{subject to} && \begin{pmatrix} \bar{\gamma}_k & f_k^T \\ f_k & K(x) \end{pmatrix} \succeq 0 \quad \forall k \quad (\leftarrow V_k \succeq 0) \\
& && Ax \leq b \quad (\leftarrow \eta \geq 0) \\
& && x_j \in [\underline{x}_j, \bar{x}_j] \quad \forall j. \quad (\leftarrow \xi \geq 0, \nu \geq 0)
\end{aligned}$$

The Lagrange multipliers for the different constraints are indicated in the parenthesis to the right in (R-SDP). The dual of the semi-definite program (R-SDP) is given by the semi-definite program (see e.g. [19])

$$\begin{aligned}
& \underset{\substack{V_k \in \mathcal{S}_+^{d+1}, \\ \eta \in \mathbb{R}_+^m, \xi, \nu \in \mathbb{R}_+^n}}{\text{maximize}} && - \sum_{k=1}^M \text{Tr} \left(\begin{pmatrix} \bar{\gamma}_k & f_k^T \\ f_k & K_0 \end{pmatrix} V_k \right) - \eta^T b - \xi^T \bar{x} + \nu^T \underline{x} \\
\text{(D-SDP)} \quad & \text{subject to} && \sum_{k=1}^M \text{Tr} \left(\begin{pmatrix} 0 & 0 \\ 0 & K_j \end{pmatrix} V_k \right) - a_j^T \eta - \xi_j + \nu_j = \rho_j \quad \forall j \\
& && V_k \succeq 0 \quad \forall k \\
& && \eta \geq 0, \xi \geq 0, \nu \geq 0.
\end{aligned}$$

We assume, for now, that the dual variables V_k for the linear matrix inequalities are rank one and given by

$$V_k := \begin{pmatrix} \alpha_k \\ z_k \end{pmatrix} \begin{pmatrix} \alpha_k & z_k^T \end{pmatrix}$$

for some $\alpha_k \in \mathbb{R}$ and $z_k \in \mathbb{R}^d$. With this assumption, the dual (D-SDP) of the semi-definite program (R-SDP), can be written as the quadratically constrained quadratic program

$$\begin{aligned}
& \underset{\substack{\alpha_k \in \mathbb{R}, z_k \in \mathbb{R}^d, \\ \eta \in \mathbb{R}_+^m, \xi, \nu \in \mathbb{R}_+^n}}{\text{maximize}} && - \sum_{k=1}^M \begin{pmatrix} \alpha_k \\ z_k \end{pmatrix}^T \begin{pmatrix} \bar{\gamma}_k & f_k^T \\ f_k & K_0 \end{pmatrix} \begin{pmatrix} \alpha_k \\ z_k \end{pmatrix} - \eta^T b - \xi^T \bar{x} + \nu^T \underline{x} \\
& \text{subject to} && \sum_{k=1}^M z_k^T K_j z_k - a_j^T \eta - \xi_j + \nu_j = \rho_j \quad \forall j \\
& && \eta \geq 0, \xi \geq 0, \nu \geq 0.
\end{aligned}$$

After eliminating the variable ν and switching from maximization to minimization in the above problem formulation we arrive at the inequality con-

strained problem

$$\begin{aligned}
& \underset{\substack{\alpha_k \in \mathbb{R}, z_k \in \mathbb{R}^d, \\ \eta \in \mathbb{R}_+^m, \xi \in \mathbb{R}_+^n}}{\text{minimize}} && \sum_{k=1}^M \begin{pmatrix} \alpha_k \\ z_k \end{pmatrix}^T \begin{pmatrix} \bar{\gamma}_k & f_k^T \\ f_k & K(\underline{x}) \end{pmatrix} \begin{pmatrix} \alpha_k \\ z_k \end{pmatrix} \\
& && -\eta^T (A\underline{x} - b) + \xi^T (\bar{x} - \underline{x}) - \rho^T \underline{x} \\
(Q2P) \quad & \text{subject to} && \sum_{k=1}^M z_k^T K_j z_k - a_j^T \eta - \xi_j \leq \rho_j \quad \forall j \\
& && \eta \geq 0, \xi \geq 0.
\end{aligned}$$

3.3 Properties of the reformulations

Some observations regarding the relaxation (R) and the reformulations (R-SDP), (D-SDP), and (Q2P) are presented below.

Remark 1. *The feasible set of the quadratically constrained quadratic program (Q2P) is nonempty since the point*

$$(\alpha_1, z_1, \dots, \alpha_M, z_M, \eta, \xi) = (0, 0, \dots, 0, 0, 0, 0)$$

is feasible to (Q2P). Furthermore, this problem satisfies Slater's constraint qualifications since the feasible set contains interior points.

Remark 2. *The feasible set of (Q2P) is convex since the (scaled) element stiffness matrices K_j are assumed to be positive semi-definite. The problem (Q2P), however, fails to be a convex problem since the objective function is not a convex function (the matrix $K(\underline{x})$ may very well be the zero matrix).*

Remark 3. *The feasible set of the dual semi-definite program (D-SDP) is also non-empty since the point*

$$(V_1, \dots, V_m, \eta, \xi, \nu) = (0, \dots, 0, 0, 0, \rho)$$

is feasible to (D-SDP). Furthermore, the problem (D-SDP) satisfies Slater's constraint qualifications.

Remark 4. *If $(\alpha_1, z_1, \dots, \alpha_M, z_M, \eta, \xi)$ is a feasible point to (Q2P) then $(V_1, \dots, V_M, \eta, \xi, \nu)$ where*

$$\begin{aligned}
V_k &:= \begin{pmatrix} \alpha_k \\ z_k \end{pmatrix} \begin{pmatrix} \alpha_k & z_k^T \end{pmatrix} \succeq 0 && k = 1, \dots, M \\
\nu_j &:= \rho_j - \left(\sum_{k=1}^M z_k^T K_j z_k - a_j^T \eta - \xi_j \right) \geq 0 && j = 1, \dots, n
\end{aligned}$$

is a feasible point to the semi-definite program (D-SDP). Because of weak duality between (R-SDP) and (D-SDP) any feasible point to (D-SDP) gives a lower bound on the optimal objective function value in (R-SDP) and hence also on the optimal objective function value in (R). Hence, any feasible point to (Q2P) gives (via (D-SDP)) a lower bound on the objective function value in (R).

Lemma 2. Let $(\alpha_1, z_1, \dots, \alpha_M, z_M, \eta, \xi)$ be a local or global minimizer of (Q2P) with $\alpha_k \neq 0$ for $k = 1, \dots, M$ and corresponding Lagrange multipliers $y \in \mathbb{R}_+^n$, $\tau \in \mathbb{R}_+^n$, and $\sigma \in \mathbb{R}_+^m$. Then (x, u_1, \dots, u_M) is optimal for (R), where

$$\begin{aligned} x &:= y + \underline{x} \\ u_k &:= -z_k/\alpha_k \quad k = 1, \dots, M. \end{aligned}$$

Proof. Let $(\alpha_1, z_1, \dots, \alpha_M, z_M, \eta, \xi)$ be optimal for (Q2P) with $\alpha_k \neq 0$ for $k = 1, \dots, M$. Then, since the Slater constraint qualifications are satisfied for (Q2P) the Karush-Kuhn-Tucker conditions hold at $(\alpha_1, z_1, \dots, \alpha_M, z_M, \eta, \xi)$, i.e., there exists vectors $y \in \mathbb{R}_+^n$, $\tau \in \mathbb{R}_+^n$, and $\sigma \in \mathbb{R}_+^m$ such that

$$\bar{\gamma}_k \alpha_k + f_k^T z_k = 0 \quad k = 1, \dots, M \quad (1)$$

$$\alpha_k f_k + K(y + \underline{x}) z_k = 0 \quad k = 1, \dots, M \quad (2)$$

$$-A(y + \underline{x}) + b - \sigma = 0 \quad (3)$$

$$\bar{x} - (y + \underline{x}) - \tau = 0 \quad (4)$$

$$\sigma_i \eta_i = 0 \quad i = 1, \dots, m \quad (5)$$

$$\tau_j \xi_j = 0 \quad j = 1, \dots, n \quad (6)$$

$$y_j \left(\sum_{k=1}^M z_k^T K_j z_k - a_j^T \eta - \xi_j - \rho_j \right) = 0 \quad j = 1, \dots, n \quad (7)$$

$$y \geq 0, \sigma \geq 0, \tau \geq 0. \quad (8)$$

Now consider (x, u_1, \dots, u_M) as in the assertion. From conditions (3) – (4) and (8) we see that x satisfies the bound constraints $\underline{x} \leq x \leq \bar{x}$ and the general linear constraints $Ax \leq b$. From (1) – (2) we see that (x, u_1, \dots, u_M) satisfies the equilibrium conditions $K(x)u_k = f_k$ and the compliance constraints $f_k^T u_k \leq \bar{\gamma}_k$ with equality. Hence, (x, u_1, \dots, u_M) is a feasible point for problem (R). Moreover, by the complementarity condition (7)

$$\begin{aligned} \rho^T x &= \rho^T \underline{x} + \rho^T y \\ &= \rho^T \underline{x} + \sum_{j=1}^n y_j \left(\sum_{k=1}^M z_k^T K_j z_k - a_j^T \eta - \xi_j \right) \\ &= \rho^T \underline{x} + \sum_{k=1}^M z_k^T K(y) z_k - y^T A^T \eta - y^T \xi \end{aligned}$$

First use the conditions (3) and (4) and then the complementarity conditions (5) and (6)

$$\begin{aligned} &= \rho^T \underline{x} + \sum_{k=1}^M z_k^T K(y) z_k + \eta^T (A \underline{x} - b) + \eta^T \sigma - \xi^T (\bar{x} - \underline{x}) + \xi^T \tau \\ &= \rho^T \underline{x} + \sum_{k=1}^M z_k^T K(y) z_k + \eta^T (A \underline{x} - b) - \xi^T (\bar{x} - \underline{x}) \end{aligned}$$

Using $z_k^T(K(y) + K(\underline{x}))z_k = -\alpha_k f_k^T z_k = \bar{\gamma}\alpha_k^T$ gives

$$= \rho^T \underline{x} + \sum_{k=1}^M \begin{pmatrix} \alpha_k \\ z_k \end{pmatrix}^T \begin{pmatrix} \bar{\gamma}_k & 0 \\ 0 & -K(\underline{x}) \end{pmatrix} \begin{pmatrix} \alpha_k \\ z_k \end{pmatrix} + \eta^T(A\underline{x} - b) - \xi^T(\bar{x} - \underline{x})$$

By adding $-2\alpha_k f_k^T z_k$ to both sides, using $-2\alpha_k f_k^T z_k = 2\bar{\gamma}\alpha_k^2$, and rearranging we arrive at

$$\rho^T x = -\sum_{k=1}^M \begin{pmatrix} \alpha_k \\ z_k \end{pmatrix}^T \begin{pmatrix} \bar{\gamma}_k & f_k^T \\ f_k & K(\underline{x}) \end{pmatrix} \begin{pmatrix} \alpha_k \\ z_k \end{pmatrix} + \eta^T(Ax - b) - \xi^T(\bar{x} - x) + \rho^T \underline{x}.$$

The right hand side is the negative of the objective function in (Q2P). Since (x, u_1, \dots, u_M) is feasible to (R), x is also feasible to (R-SDP) with the same objective function value $\rho^T x$. From a KKT-point to (Q2P) we can construct a feasible point to the semi-definite program (D-SDP) (see Remark 4). Since the objective function values of the pair of feasible points to (R-SDP) and (D-SDP) have zero duality gap, the points are also optimal to (R-SDP) and (D-SDP), respectively. It follows from Lemma 1 that (x, u_1, \dots, u_M) is indeed an optimal solution to (R). \square

In a branch and bound method for solving the mixed integer problem (P) we solve a sequence of relaxations (R) for different values on the lower \underline{x} and upper bounds \bar{x} on the variables x . These bounds appear in the feasible set of (R) but only appear in the objective function in (Q2P) and the feasible set of (Q2P) remains the same for all choices of \underline{x} and \bar{x} . Following Remark 1 and Remark 4 a feasible point to (Q2P) and a lower bound on the objective function value in (R) are immediately available. In a branch and bound framework a good feasible point to (Q2P) is also available as the optimal solution to the relaxation in the father node (of the node under consideration).

If, for some load case(s), $\alpha_k = 0$ then we cannot use the substitution $u_k = -z_k/\alpha_k$ to construct a feasible solution (x, u_1, \dots, u_M) to the continuous relaxation (R). This is indeed theoretically unsatisfactory, but in practice and seen from the view point of branch and bound this is a minor obstacle. We only need a lower bound on the objective function value in (R) and a vector x of design variables satisfying the linear constraints in (R) such that we can continue branching. This is made more precise in the following Corollary.

Corollary 1. *Let $(\alpha_1, z_1, \dots, \alpha_M, z_M, \eta, \xi)$ be a Karush-Kuhn-Tucker point of (Q2P) with corresponding Lagrange multipliers $y \in \mathbb{R}_+^n$, $\tau \in \mathbb{R}_+^n$, and $\sigma \in \mathbb{R}_+^m$. Then x is optimal for (R-SDP) and $(V_1, \dots, V_M, \eta, \xi, \nu)$ is optimal for (D-SDP), where*

$$x := y + \underline{x}$$

and

$$V_k := \begin{pmatrix} \alpha_k \\ z_k \end{pmatrix} \begin{pmatrix} \alpha_k & z_k^T \end{pmatrix} \quad k = 1, \dots, M$$

$$\nu_j := \rho_j - \left(\sum_{k=1}^M z_k^T K_j z_k - a_j^T \eta - \xi_j \right) \quad j = 1, \dots, n.$$

3.4 Brief summary and motivation

We propose that the quadratically constrained quadratic program (Q2P) is used in a branch and bound framework to obtain lower bounds on the objective function value of the discrete problem (P). This problem has indeed the theoretical discrepancy of not being convex. The convexity properties of the semi-definite programs (R-SDP) and (D-SDP) favor the use of these problem formulations instead. However, the non-convexity is mild in the sense that finding KKT points of (Q2P) gives us an optimal solution to the convex problems (R-SDP) and (D-SDP) and hence lower bounds on the objective function value in (P). Furthermore, good feasible points will be available in a branch and bound framework and there exist methods which can utilize the available "warm-start" information when attacking (Q2P). Good feasible solutions are indeed available also for the dual (D-SDP) but, so far, the interior point methods developed for semi-definite programming cannot be efficiently "warm started". Finally, due to the close relation with the dual (D-SDP), the all-quadratic program (Q2P) must not be solved to optimality at every node in the search tree, since any feasible point to (Q2P) (or (D-SDP)) gives a lower bound. This is in sharp contrast to a branch and bound method based on solving a sequence of the primal problems (R-SDP).

4 Convex reformulation and continuous relaxation

The second class of relaxations of the discrete problem (P) is obtained by first reformulating the problem into an equivalent convex mixed 0–1 program and then relaxing the integer constraints. The reformulation requires additional assumptions on the element stiffness matrices and a more precise definition of the stress constraints. The relaxation is a continuous convex quadratically constrained linear program. Since this relaxation is significantly larger than the original problem several alternative weaker relaxations capturing the most important features are presented. For these relaxations we propose that a dual formulation suitable for use within an implementation of a branch and bound method for solving the discrete problem (P).

4.1 Statement of the second relaxation

The reformulation of the discrete problem (P) follows the approaches outlined in [13], [16], and [17] and is therefore only briefly presented. The

reformulation requires a more precise definition of the element stiffness matrices K_j and the stress constraints, i.e., the matrices W_j . In the following we assume that the (scaled) element stiffness matrices are given as

$$K_j = B_j E_j B_j^T \quad (\text{A9})$$

where $B_j^T \in \mathbb{R}^{r_j \times d}$ is the (scaled) strain-displacement matrix and $E_j = E_j^T \in \mathcal{S}_+^{r_j}$ is a block-diagonal matrix containing the material stiffness. The first part of the reformulation of (P) is the introduction of additional force-like continuous variables $\mathbb{R}^{r_j} \ni q_j = x_j E_j B_j^T u$ for $j = 1, \dots, n$. For simplicity we throughout this section assume that only a single load is applied, and hence remove the index k from all formulations. With these continuous variables, the equilibrium equations $K(x)u = f$ are disaggregated into the linear equality

$$\sum_{j=1}^n B_j q_j = f$$

which express node equilibrium of forces and the bilinear equalities

$$q_j = x_j E_j B_j^T u \quad \forall j = 1, \dots, n.$$

The above bilinear equalities are equivalently written as the set of linear inequalities

$$\begin{aligned} x_j \underline{c} &\leq q_j \leq x_j \bar{c} && \forall j \\ (1 - x_j) \underline{c} &\leq E_j B_j^T u - q_j \leq (1 - x_j) \bar{c} && \forall j \end{aligned} \quad (9)$$

where the constant vectors $\underline{c} \in \mathbb{R}^{r_j}$ and $\bar{c} \in \mathbb{R}^{r_j}$ are given by

$$\underline{c} = (E_j B_j^T)^+ \underline{u} + (E_j B_j^T)^- \bar{u}$$

and

$$\bar{c} = (E_j B_j^T)^- \underline{u} + (E_j B_j^T)^+ \bar{u}.$$

With the force-like variables q_j we assume that the stress constraints can be expressed as

$$q_j^T \widehat{V}_j^T \widehat{V}_j q_j \leq x_j \bar{\sigma}^2 \quad \forall j = 1, \dots, n \quad (\text{A10})$$

for some matrix $\widehat{V}_j \in \mathbb{R}^{r_j \times r_j}$. Notice that the stress constraints, when formulated in the force-like variables q_j , are included for all elements, irrespective of the value on the design variables. With these observations, the original problem can be equivalently reformulated as a problem with linear and convex quadratic inequality constraints. The natural continuous

relaxation of this equivalent mixed integer problem is given by

$$\begin{aligned}
& \underset{\substack{x \in \mathbb{R}^n, u \in \mathbb{R}^d \\ q_1 \in \mathbb{R}^{r_1}, \dots, q_n \in \mathbb{R}^{r_n}}}{\text{minimize}} && \rho^T x \\
\text{(R2)} \quad & \text{subject to} && \sum_{j=1}^n B_j q_j = f \\
& && x_j \underline{c} \leq q_j \leq x_j \bar{c} && \forall j \\
& && E_j B_j^T u - q_j \geq (1 - x_j) \underline{c} && \forall j \\
& && E_j B_j^T u - q_j \leq (1 - x_j) \bar{c} && \forall j \\
& && f^T u \leq \bar{\gamma} \\
& && q_j^T \widehat{V}_j^T \widehat{V}_j q_j \leq x_j \bar{\sigma}^2 && \forall j \\
& && \underline{u} \leq u \leq \bar{u} \\
& && Ax \leq b \\
& && x_j \in [\underline{x}_j, \bar{x}_j] && \forall j.
\end{aligned}$$

The relaxation (R2) is much larger in size than the relaxation (R) both in terms of number of variables and constraints. The relaxation (R2) however includes the stress and displacement constraints and it is a convex problem.

4.2 Condensing the second relaxation

The relaxation (R2) can, on its own merit, be used to design a finitely convergent branch and bound method for solving the mixed integer program (P). The reformulation and the continuous relaxation (R2), although appealing in theory, suffers from two serious deficiencies. Firstly, the strength of this relaxation is severely hampered by the reformulation of the bilinear equalities. This is essentially a "big-M" construction which is well-known in the integer programming community to produce weak relaxations. The preliminary numerical results in [4] and [16] for truss topology optimization problems clearly support this observation. Secondly, as already mentioned, the relaxation (R2) is much larger in size than the relaxation (R) and the original problem (P) both in terms of number of variables and constraints. The feasible set of the relaxation (R2) is also dependent on the bounds on the design variables and hence, in a branch and bound context on previous branching decisions. Therefore, feasible points are not immediately available for (R2) in a branch and bound framework. Because of the number of constraints in (R2) a dual formulation of (R2) will also become a large-scale problem.

We therefore propose to decrease the size of the relaxation (R2) by removing the weak constraints (9) while maintaining the property that the relaxation should be sufficiently strong to produce a convergent branch and bound method. This is accomplished by changing the relaxation depending on the position in a branch and bound search tree. This approach leads to

another, weaker, relaxation which hopefully is much faster to solve in practice. This condensed relaxation is then strengthened by adding some trivial valid inequalities, rewriting the compliance as a nonlinear convex constraint, and strengthening the stress constraints. This strengthening procedure can also be applied to the relaxation (R2).

Introduce the index sets $\mathcal{N} = \{1, \dots, n\}$, $\mathcal{N}_0 = \{j \in \mathcal{N} \mid \underline{x}_j = \bar{x}_j = 0\}$, and $\mathcal{N}_1 = \{j \in \mathcal{N} \mid \underline{x}_j = \bar{x}_j = 1\}$, i.e., the set of all design variables and the sets of design variables fixed to zero and one, respectively. The compliance $f^T u$ can equivalently (for all (x, u) feasible to (P)) be written as

$$f^T u = u^T K(x) u = \sum_{j \in \mathcal{N}} x_j u^T K_j u = \sum_{j \in \mathcal{N}_1} u^T K_j u + \sum_{j \in \mathcal{N} \setminus (\mathcal{N}_0 \cup \mathcal{N}_1)} \frac{q_j^T C_j q_j}{x_j}$$

where $C_j = E_j^{-1}$ and we interpret $q_j^T C_j q_j / x_j = 0$ if $x_j = 0$. The linear compliance constraint $f^T u \leq \bar{\gamma}$ can thus be supplemented with the constraints

$$\begin{aligned} \sum_{j \in \mathcal{N}} t_j &\leq \bar{\gamma} \\ q_j^T C_j q_j &\leq x_j t_j \quad j \in \mathcal{N} \setminus (\mathcal{N}_0 \cup \mathcal{N}_1) \\ u^T K_j u &\leq t_j \quad j \in \mathcal{N}_1 \\ t_j &\geq 0 \quad j \in \mathcal{N} \end{aligned}$$

where $t \in \mathbb{R}_+^n$ is a set of additional continuous variables. It also follows that compliance is always positive and we can add the linear valid inequality

$$f^T u \geq 0$$

to the problem formulation. The final point on this agenda is a strengthening of the stress constraints. Since $x_j = x_j^2$ for $x_j \in \{0, 1\}$ the stress constraints can be written as

$$q_j^T \widehat{V}_j^T \widehat{V}_j q_j \leq x_j^2 \bar{\sigma}^2 \quad j \in \mathcal{N} \setminus (\mathcal{N}_0 \cup \mathcal{N}_1).$$

Now the right hand side goes as rapidly to zero as the left hand side. For those design variables which are fixed to one, we rewrite the stress constraints in the displacement variables, i.e.,

$$u^T W_j u \leq \bar{\sigma}^2 \quad j \in \mathcal{N}_1.$$

The linear equality constraint can equivalently be written as

$$K(\underline{x}) + Bq = f$$

where B is a sub-matrix of the matrix $(B_1 \ B_2 \ \dots \ B_n)$, i.e., $B = (B_j)$ for $j \in \mathcal{N} \setminus (\mathcal{N}_0 \cup \mathcal{N}_1)$.

We arrive at the condensed and partially strengthened relaxation

$$\begin{aligned}
& \text{minimize} && \rho^T x \\
& \begin{array}{l} x \in \mathbb{R}^n, u \in \mathbb{R}^d \\ q_j \in \mathbb{R}^{r_j}, t \in \mathbb{R}^n \end{array} && \\
\text{(R3)} \quad & \text{subject to} && K(x) + Bq = f && (\leftarrow \nu) \\
& && 0 \leq f^T u \leq \bar{\gamma} && (\leftarrow \delta, \alpha) \\
& && \sum_{j \in \mathcal{N}} t_j \leq \bar{\gamma} && (\leftarrow \delta) \\
& && q_j^T C_j q_j \leq x_j t_j && j \in \mathcal{N} \setminus (\mathcal{N}_0 \cup \mathcal{N}_1) \quad (\leftarrow \pi_j) \\
& && u^T K_j u \leq t_j && j \in \mathcal{N}_1 \quad (\leftarrow \pi_j) \\
& && q_j^T \widehat{V}_j^T \widehat{V}_j q_j \leq x_j^2 \bar{\sigma}^2 && j \in \mathcal{N} \setminus (\mathcal{N}_0 \cup \mathcal{N}_1) \quad (\leftarrow \lambda_j) \\
& && u^T W_j u \leq \bar{\sigma}^2 && j \in \mathcal{N}_1 \quad (\leftarrow \lambda_j) \\
& && Ax \leq b && (\leftarrow \eta) \\
& && \underline{u} \leq u \leq \bar{u} && (\leftarrow \xi_u, \nu_u) \\
& && t \geq 0 && (\leftarrow \tau) \\
& && x_j \in [\underline{x}_j, \bar{x}_j] && \forall j. \quad (\leftarrow \xi_x, \nu_x)
\end{aligned}$$

Although it may not be apparent at first sight, the relaxation (R3) contains significantly fewer linear constraints than (R2). Furthermore, in this relaxation we have eliminated the force-like variables and stress constraints corresponding to design variables which are fixed. Hence the size of the subproblem (both in terms of variables and constraints) will decrease with increasing depth in a branch and bound search tree. Notice that we, for simplicity, do not eliminate the corresponding t_j although this is an option. Again, the relaxation is an all-quadratic program.

4.3 Dual of the second relaxation

In this section we formulate a dual of the all-quadratic relaxation (R3) using conic duality. First, the quadratic constraints are rewritten as second order cone constraints. The reformulation techniques and the duality results are based on the presentation in [11]. We make the assumptions that the matrix E_j is decomposed as $E_j = \widehat{E}_j \widehat{E}_j^T$ and that C_j is decomposed as $C_j = \widehat{C}_j^T \widehat{C}_j$. This implies that the (scaled) element stiffness matrix

$$K_j = B_j E_j B_j^T = B_j \widehat{E}_j \widehat{E}_j^T B_j^T.$$

The energy strain constraints $u^T K_j u \leq t_j$ are reformulated as

$$\left\| \begin{pmatrix} \widehat{E}_j^T B_j^T u \\ (t_j - 1)/2 \end{pmatrix} \right\|_2 \leq \frac{t_j + 1}{2} \quad j \in \mathcal{N}_1.$$

The constraints $q_j^T C_j q_j \leq x_j t_j$ with $x_j \geq 0$ and $t_j \geq 0$ are formulated as

$$\left\| \begin{pmatrix} \widehat{C}_j q_j \\ (x_j - t_j)/2 \end{pmatrix} \right\|_2 \leq \frac{x_j + t_j}{2} \quad j \in \mathcal{N} \setminus (\mathcal{N}_0 \cup \mathcal{N}_1).$$

The stress constraints, for design variables which have been fixed to one, are given by

$$u^T W_j u = u^T \widehat{W}_j^T \widehat{W}_j u \leq \bar{\sigma}^2 \quad j \in \mathcal{N}_1.$$

This quadratic constraint is equivalent to the second order cone constraint

$$\|\widehat{W}_j u\|_2 \leq \bar{\sigma} \quad j \in \mathcal{N}_1.$$

Finally, the stress constraints $q_j^T \widehat{V}_j^T \widehat{V}_j q_j \leq x_j^2 \bar{\sigma}^2$ are written as

$$\|\widehat{V}_j q_j\|_2 \leq \bar{\sigma} x_j \quad j \in \mathcal{N} \setminus (\mathcal{N}_0 \cup \mathcal{N}_1).$$

With these reformulations we can transform (identify $z = (x, u, q, t)$ and $c = (\rho, 0, 0, 0)$) the relaxation (R3) into a second order cone program on the following form

$$\begin{aligned} & \underset{z}{\text{minimize}} && c^T z \\ \text{(SOCP)} & \text{subject to} && Pz = p && (\leftarrow \nu) \\ & && Qz \geq q && (\leftarrow \eta \geq 0) \\ & && G_j z - g_j \succeq_{L^{r_j+1}} 0 \quad j \in \mathcal{N} \setminus \mathcal{N}_0 && (\leftarrow \lambda_j \succeq_{L^{r_j+1}} 0) \\ & && H_j z - h_j \succeq_{L^{r_j+1}} 0 \quad j \in \mathcal{N} \setminus \mathcal{N}_0 && (\leftarrow \pi_j \succeq_{L^{r_j+1}} 0). \end{aligned}$$

Here, L^{r_j+1} denote the Lorentz (or the second order, or the ice cream) cone

$$L^{r_j+1} = \{\lambda = (\lambda_1^1, \dots, \lambda_{r_j}^1, \lambda^0) = (\lambda^1, \lambda^0) \in \mathbb{R}^{r_j+1} \mid \lambda^0 \geq \|\lambda^1\|_2\}.$$

The conic dual of (SOCP) is given by

$$\begin{aligned} & \underset{\nu, \eta, \lambda_j, \pi_j}{\text{maximize}} && p^T \nu + q^T \eta + \sum_{j \in \mathcal{N} \setminus \mathcal{N}_0} (\lambda_j^T g_j + \pi_j^T h_j) \\ \text{(D-SOCP)} & \text{subject to} && P^T \nu + Q^T \eta + \sum_{j \in \mathcal{N} \setminus \mathcal{N}_0} (G_j^T \lambda_j + H_j^T \pi_j) = c \\ & && \lambda_j \succeq_{L^{r_j+1}} 0, \quad j \in \mathcal{N} \setminus \mathcal{N}_0 \\ & && \pi_j \succeq_{L^{r_j+1}} 0, \quad j \in \mathcal{N} \setminus \mathcal{N}_0 \\ & && \eta \geq 0. \end{aligned}$$

The conic dual of the all-quadratic relaxation (R3) is (after tedious reformulations) finally given by

$$\begin{aligned}
\max \quad & -\eta^T b + \xi_x^T \underline{x} + \nu_x^T \bar{x} - (\alpha + \beta)\bar{\gamma} + \xi_u^T \underline{u} - \nu_u^T \bar{u} + \nu^T f \\
& -\bar{\sigma} \sum_{j \in \mathcal{N}_1} \lambda_j^0 + \frac{1}{2} \sum_{j \in \mathcal{N}_1} (e_{r_j}^T \pi_j - \pi_j^0) \\
\text{(D-R3)} \quad \text{s.t.} \quad & -A^T \eta + \xi_x - \nu_x + \sum_{j \in \mathcal{N} \setminus (\mathcal{N}_0 \cup \mathcal{N}_1)} (\bar{\sigma} \lambda_j^0 + \frac{1}{2} \pi_j^0) e_j = \rho \\
& K(\underline{x})\nu + (\delta - \alpha)f + \xi_u - \nu_u + \sum_{j \in \mathcal{N}_1} (\widehat{W}_j^T \lambda_j^1 + B_j^T \widehat{E}_j \pi_j^1) = 0 \\
& B^T \nu + \sum_{j \in \mathcal{N} \setminus (\mathcal{N}_0 \cup \mathcal{N}_1)} (\widehat{V}_j \lambda_j^1 + \widehat{C}_j^T \pi_j^1) = 0 \\
& -\beta e + \tau + \frac{1}{2} \sum_{j \in \mathcal{N}_1} \pi_j^0 e_j - \frac{1}{2} \sum_{j \in \mathcal{N} \setminus (\mathcal{N}_0 \cup \mathcal{N}_1)} \pi_j^0 e_j = 0 \\
& \lambda_j \succeq_{L^{r_j+1}} 0, \quad j \in \mathcal{N} \setminus \mathcal{N}_0 \\
& \pi_j \succeq_{L^{r_j+1}} 0, \quad j \in \mathcal{N} \setminus \mathcal{N}_0 \\
& \alpha, \delta, \eta, \xi_x, \nu_x, \xi_u, \nu_u, \tau \geq 0.
\end{aligned}$$

Remark 5. *The feasible set of the second order cone program (D-R3) is nonempty since the point with $\xi_x := \rho$ and all the other variables equal to zero is feasible.*

Remark 6. *Let $y \in \mathbb{R}^n$ denote the Lagrange multipliers for the first linear equality constraint in (D-R3) and let $w \in \mathbb{R}^d$ denote the Lagrange multiplier for the second equality constraint in (D-R3). If an optimal solution of (D-R3) is found then (x, u) where*

$$\begin{aligned}
x & := -y \\
u & := -w
\end{aligned}$$

satisfies $\underline{x} \leq x \leq \bar{x}$ and $Ax \leq b$, and $\underline{u} \leq u \leq \bar{u}$. Hence, this point is eligible for further branching.

Remark 7. *If a dedicated solver for second order cone programs is not available, then the problem (D-R3) can be reformulated as an all-quadratic separable program by using the matrix $Q = \text{diag}(1, \dots, 1, -1) \in \mathbb{R}^{r_j+1 \times r_j+1}$ and rewriting the constraints*

$$\begin{aligned}
\lambda_j & \succeq_{L^{r_j+1}} 0 \quad j \in \mathcal{N} \setminus \mathcal{N}_0 \\
\pi_j & \succeq_{L^{r_j+1}} 0 \quad j \in \mathcal{N} \setminus \mathcal{N}_0
\end{aligned}$$

as

$$\begin{aligned}
\lambda_j^T Q \lambda & \leq 0 \quad \lambda_j^0 \geq 0 \quad j \in \mathcal{N} \setminus \mathcal{N}_0 \\
\pi_j^T Q \pi_j & \leq 0 \quad \pi_j^0 \geq 0 \quad j \in \mathcal{N} \setminus \mathcal{N}_0.
\end{aligned}$$

The all-quadratic reformulation of the second order cone program becomes non-convex. However, since any feasible point to (D-R3) provides a lower bound to the original discrete problem (P) it is not necessary to solve non-convex problem to global optimality.

5 Summary and outline of future research

We present several convex and non-convex continuous relaxations of a particular class of discrete structural topology optimization problems with stress constraints. These relaxations are reformulated using duality results from semi-definite and second order cone programming. The reformulated problems are suitable for implementation in a nonlinear branch and bound framework for solving the considered class of problems to guaranteed global optimality.

The strength of the proposed lower bounding problems will be numerically investigated on several classical and new benchmark examples from the field of structural topology optimization. A nonlinear and convergent branch and bound method based on combining these relaxations will be implemented and numerically evaluated on a large set of medium to large-scale benchmark examples. The implementation details and the numerical investigations are the subjects of forthcoming manuscripts.

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