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Free material optimization with multidisciplinary optimization constraints for plates and shells

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FREE MATERIAL OPTIMIZATION WITH MULTIDISCIPLINARY OPTIMIZATION CONSTRAINTS FOR PLATES AND SHELLS *

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Abstract

Free Material Optimization (FMO) is a branch of structural optimization and deals with the problem of finding the stiffest structure for a given design domain and a given set of loads, where the amount of available material is limited. The choice of material in FMO is not restricted to a certain material symmetry or to material that already exists in nature. Instead the entire material tensor is taken as design variable yielding not only the optimal material distribution, but also the optimal material properties at each point of the design domain. The FMO formalism has been extended to Naghdi shells and Reissner-Mindlin plates in previous contributions of the authors. In this article we consider additional constraints on the FMO problem for shells – namely displacement, stress, vibration and global stability constraints – to find a design more suited to realistic problems and their requirements.

Keywords: Free Material Optimization, Naghdi Shells, Displacement Constraints, Stress Constraints, Vibration Constraints, Global Stability Constraints.

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1. Introduction

Over the last decades the usage of so-called advanced materials has become commonplace in structural engineering. Advanced materials are manmade materials, which are not preexistent in nature. They can be constructed by laminating layers with different thickness and different material together, or even by constructing laminates of a higher rank from different materials. Another possibility would be to create foams of a custom density. All these procedures lead to new materials thus providing an infinite number of materials to choose from when designing structures. In this wide range of possibilities naturally the question arises which of all possible materials is the material best suited to the application at hand?

The problem of finding the optimal material for a given structural design problem has established the field of material optimization, which intends to find optimal material parameters using mathematical optimization methods. In this field of research there exist various procedures differing in the choice of admissible materials. It is for example possible to do material optimization when only using one fixed material [23]. By varying the density of this material (for example by constructing foams with different size and number of holes) different material tensors are accessible (although the material symmetry and orientation stay constant). Solutions for this problem can be obtained by using a SIMP-approach without penalization. A very traditional approach in material optimization is the inclusion of stiffeners and fillets as described in [21, 34]. The determination of the right size, shape and material of these stiffeners lead to completely new material properties of the designed structure. Even more freedom in the material design space is obtained by optimizing the layer thicknesses and fibre orientation of orthotropic laminates [29]. The approach of optimizing fibre angles was picked up in the field of discrete material optimization. Dealing also with the optimal construction of laminated composites it defines a set of predefined materials. During optimization the optimal material from this set together with its fibre orientation is determined [35]. [3] proposed to optimize not over a limited set of materials, but instead over all possible material tensors. Thus the complete freedom in the material space is exploited although the optimal materials might not preexist in nature. This approach is known as Free Material Optimization.

Free Material Optimization deals with one of the basic problems in structural optimization – finding the lightest structure in a given design domain, that is able to carry a predefined set of loads. As introduced by [3] the design variable in Free Material Optimization is the full elasticity tensor thereby granting complete freedom in the choice of the optimal material. Thus not only the optimal material distribution is provided in the optimum, but also the optimal material properties at each point of the design domain. Usually there will be no natural material fitting the optimal material properties, but the effort of constructing e.g. laminates with the given material tensors is worthwhile in fields like aerodynamics where there is a high need of light, yet stable structures. Problem formulation and analysis were performed in [38, 20], where it was also shown that the Free Material Optimization problem is tractable by mathematical programming methods. [24] proved existence of a solution to the Free Material Optimization problem in a saddlepoint formulation and convergence of the discretized problem formulation using finite elements. This was also done by [37], who demonstrated how to use Lagrange duality theory to transform the saddlepoint formulation into a linear quadratically constrained optimization problem in the single load case. When multiple loads are applied this method leads to a convex nonlinear semidefinite programm (SDP) [1]. The discretized formulation of this problem is e.g. solvable by the nonlinear SDP code PENNON [16]. As many applications of Free Material Optimization lie in the field of aerodynamics it was promising to extend the formulation to thin-walled structures like plates and shells to be able to optimize the material of thin parts like e.g. the fuselage. In recent publications [12, 13] the authors developed ideas of Free Material Optimization for Reissner-Mindlin plates presented in [2] further and found a formulation of the Free Material Optimization problem for shells. They showed existence of solutions and equivalence of the minimum compliance formulation to a nonlinear convex SDP.

This article is concerned with the extension of the Free Material Optimization problem for Naghdi shells by multidisciplinary optimization constraints. First of all we consider linear displacement constraints as presented in [19] to prescribe shapes of the deformed shell and even to construct mechanical mechanisms. Next we tend to stress constraints to avoid material failure due to high stresses. While the obtained optimization problem is suited for the same optimization algorithms as the stress constrained problems presented in [18] we have to distinguish in-plane and out-of-plane stresses in the problem formulation for shells. Another interesting type of constraints are eigenfrequency constraints. Constraints of this type have already been considered when optimizing fiber reinforced plates [28] and also in the context of Free Material Optimization for solids [36]. We use the methods presented in [4] to obtain the equilibrium problem for free vibrations of Naghdi shells and are thus able to formulate the Free Material Optimization problem for shells with constraints on the fundamental eigenfrequency to obtain structures less susceptible to vibration resonance. Finally stability constraints are taken into account, as structural failure due to buckling is of high significance in the case of shells [6]. Based on a nonlinear extension of Naghdi's shell model we investigate the stability of the static equilibrium to predict buckling behaviour as in [4, 25, 31]. This leads to a SDP problem with nonlinear matrix constraints analogous to [17].

2. Free Material Optimization for Plates and Shells

2.1. Differential Geometry

A shell is a three-dimensional body that is thin in one direction. Thus the shell's geometry can be defined by its midsurface $\mathcal{S}$ using curvilinear coordinates. Let the reference domain $\omega \subset \mathbb{R}^2$ be open and bounded and denote by $\Phi : \mathbb{R}^2 \to \mathbb{R}^3$ a sufficiently smooth mapping from $\overline{\omega}$ into the physical space $\mathcal{P}$, e.g. $\Phi \in W^{2,\infty}(\omega)$, such that $\mathcal{S} = \Phi(\overline{\omega})$ [7]. The curvilinear coordinates are labelled by ξ_i with $i \in \{1, 2, 3\}$. In general we assume that latin indices run over 1, 2 and 3, while greek indices take the values 1 and 2.

The covariant basis vectors on the midsurface are given by

$$a_\alpha = \frac{\partial \Phi}{\partial \xi^\alpha} \quad , \quad a_3 = \frac{a_1 \times a_2}{\|a_1 \times a_2\|} \tag{1}$$

leading to the first fundamental form

$$a_{\alpha\beta} = a_\alpha \cdot a_\beta \tag{2}$$

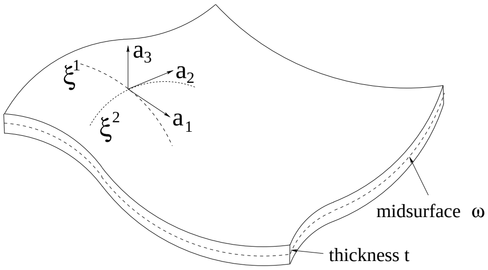

Fig. 1: Covariant basis vectors

of the midsurface. Its determinant

$$a = \det\left(a_{\alpha\beta}\right) = a_{11}a_{22} - {a_{12}}^2 \tag{3}$$

links the differentials $d\xi^1$ and $d\xi^2$ to the infinitesimal area $d\mathcal{S}$ appearing in surface integrals

$$d\mathcal{S} = \|a_1 \times a_2\|\, d\xi^1 d\xi^2 = \sqrt{\det\left(a_{\alpha\beta}\right)}\, d\xi^1 d\xi^2 = \sqrt{a}\, d\xi^1 d\xi^2 \,. \tag{4}$$

Defining by $v_{\alpha,\mu}$ the partial derivative of v_α with respect to ξ^μ for a vector field v and by

$$\Gamma^\lambda_{\alpha\mu} = a_{\alpha,\mu} \cdot a^\lambda \tag{5}$$

the Christoffel symbol on the midsurface the surface covariant derivative of v_α can be written as

$$v_{\alpha|\mu} = v_{\alpha,\mu} - \Gamma^\lambda_{\alpha\mu} v_\lambda \,. \tag{6}$$

Using this it is possible to introduce the second fundamental forms

$$\begin{aligned} b_{\alpha\beta} &= -a_{3,\beta} \cdot a_\alpha \\ b^\alpha_\beta &= -a_{3,\beta} \cdot a^\alpha \end{aligned} \tag{7}$$

and third fundamental form of the midsurface

$$c_{\alpha\beta} = b^\lambda_\alpha b_{\lambda\beta} \,. \tag{8}$$

The second fundamental form contains information about the curvature of the midsurface and can be used to calculate the mean curvature

$$\mathfrak{H} = \frac{1}{2} \operatorname{tr}\left(b^\alpha_\beta\right) = \frac{1}{2}\left(b^1_1 + b^2_2\right) \tag{9}$$

and the Gaussian (or total) curvature of the midsurface

$$\mathfrak{K} = \det\left(b^\alpha_\beta\right) = b^1_1 b^2_2 - b^1_2 b^2_1 \,. \tag{10}$$

The importance of these curvatures lies in the connection between surface integrals and three-dimensional volume integrals. For a given metric tensor g_{ij} of the three-dimensional space the infinitesimal volume $d\mathcal{R}$ is given by

$$d\mathcal{R} = \sqrt{\det\left(g_{ij}\right)}\, d\xi^1 d\xi^2 d\xi^3 = \sqrt{g}\, d\xi^1 d\xi^2 d\xi^3 \,. \tag{11}$$

As the first fundamental form $a_{\alpha\beta}$ can be interpreted as the restriction of g_{ij} to the tangent plane their determinants are related via

$$g = a\left(1 - 2\mathfrak{H}\,\xi^3 + \mathfrak{K}\left(\xi^3\right)^2\right)^2 . \tag{12}$$

A proof of this formula can for example be found in [7, Section 2.2.3, Proof of (2.155)].

2.2. Naghdi's shell model

While the midsurface is sufficient to describe the shell's geometry, it contains not enough information to properly model the physical behaviour of the shell body. Cosserat continua approach this problem by attaching a director vector d to each point $x \in \omega$ of the midsurface [9, 33]. These director vectors can be interpreted as material lines along the shell's thickness t and suffice to describe the displacements of all points in the entire three-dimensional shell body $\mathcal{R}$. Yet, a Cosserat shell refers to the shell as a two-dimensional object and thus offers the opportunity to capture a shell's unique behaviour. When considering a loaded Cosserat shell the displacements can be separated into two parts: the translational displacements $u \in \left[H^1(\omega)\right]^3$ of all points on the midsurface and the rotational displacements of the attached director vectors. As the associated material lines are regarded to be infinitely thin rotations of the director vectors around their own axis can be neglected. Thus the rotational displacements are defined by the rotation vector $\theta \in \left[H^1(\omega)\right]^2$ normal to the material lines.

The next step consists of choosing an appropriate shell model. Throughout this article we will focus on Naghdi shells, a first order approximation including shear effects [26, 7, 4, 8]. In the scope of this model the displacements take the following form:

$$U(\xi^1,\xi^2,\xi^3) = u(\xi^1,\xi^2) + \xi^3\theta_\lambda(\xi^1,\xi^2)a^\lambda(\xi^1,\xi^2)\,. \tag{13}$$

Furthermore it is assumed that the shell has a Lipschitz boundary $\partial\omega$. This boundary $\partial\omega$ is separated into a set $\partial\omega_0$, where the shell is clamped resulting in Dirichlet boundary conditions, and the remainder $\partial\omega_1$, where the shell can move freely and hence forces and moments can be applied. Both sets $\partial\omega_0$ and $\partial\omega_1$ are open in $\partial\omega$ and it holds that $\partial\omega = \overline{\partial\omega_0} \cup \overline{\partial\omega_1}$ and $\partial\omega_0 \cap \partial\omega_1 = \emptyset$. With these definitions the set of admissible displacements can be written as

$$\begin{aligned}\mathcal{U} \;:=\; \Big\{(u,\theta) \in \left[H^1(\omega)\right]^5 \,\Big|\, &\exists\, i \in \{1,2,3\} \text{ and/or } \alpha \in \{1,2\} \text{ such that} \\ &u_i = 0 \text{ and/or } \theta_\alpha = 0 \text{ on } \partial\omega_0\Big\}.\end{aligned} \tag{14}$$

Obviously it holds that $\left[H^1_0(\omega)\right]^5 \subset \mathcal{U} \subset \left[H^1(\omega)\right]^5$. The displacements of the form (13) lead to the following formulas for the membrane strain $\gamma_{\alpha\beta}$, bending strain $\chi_{\alpha\beta}$ and shear strain ζ_α, respectively:

$$\begin{aligned}
\gamma_{\alpha\beta}(u) &= \frac{1}{2}\left(u_{\alpha|\beta} + u_{\beta|\alpha}\right) - b_{\alpha\beta}u_3\,, && (15)\\
\chi_{\alpha\beta}(u,\theta) &= \frac{1}{2}\left(\theta_{\alpha|\beta} + \theta_{\beta|\alpha} - b^\lambda_\beta u_{\lambda|\alpha} - b^\lambda_\alpha u_{\lambda|\beta}\right) + c_{\alpha\beta}u_3\,, && (16)\\
\zeta_\alpha(u,\theta) &= \frac{1}{2}\left(\theta_\alpha + u_{3,\alpha} + b^\lambda_\alpha u_\lambda\right)\,. && (17)
\end{aligned}$$

Furthermore Naghdi's shell model is a linear elastic model, thus its Hooke's law takes the form

$$\begin{aligned}
N^{\lambda\mu} &= t\,C^{\lambda\mu\alpha\beta}\gamma_{\alpha\beta}\,,\\
M^{\lambda\mu} &= \frac{t^3}{12}\,C^{\lambda\mu\alpha\beta}\chi_{\alpha\beta}\,,\\
m^\lambda &= t\,k\,D^{\lambda\alpha}\zeta_\alpha\,.
\end{aligned} \tag{18}$$

The fourth-order tensor $C^{\lambda\mu\alpha\beta}$ satisfying

$$C^{\lambda\mu\alpha\beta} = C^{\mu\lambda\alpha\beta} \quad , \quad C^{\lambda\mu\alpha\beta} = C^{\lambda\mu\beta\alpha} \quad , \quad C^{\lambda\mu\alpha\beta} = C^{\alpha\beta\lambda\mu} \tag{19}$$

and the second-order tensor $D^{\lambda\alpha}$ with $D^{\lambda\alpha} = D^{\alpha\lambda}$ are the elasticity tensors of the shell. $t = t(x) \geq 0$ denotes the thickness profile of the shell which is not subject to change during the optimization process. The shear correction factor is given by $k \in [0,1] \subset \mathbb{R}_+$. Its exact value is only known for some special cases like isotropic material ($k = \frac{5}{6}$) or orthotropic material ($k = \frac{2}{3}$). The quantities appearing on the left hand side of equation (18) are the force resultant $N^{\lambda\mu}$, the moment resultant $M^{\lambda\mu}$ and the transverse shear force resultant m^λ. As shown in [30] the symmetry of the tensors can be used to rewrite Hooke's law as a linear vector equation including

$$\gamma = \begin{pmatrix} \gamma_{11} \\ \gamma_{22} \\ \sqrt{2}\gamma_{12} \end{pmatrix}, \qquad \chi = \begin{pmatrix} \chi_{11} \\ \chi_{22} \\ \sqrt{2}\chi_{12} \end{pmatrix}, \qquad \zeta = \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix}, \tag{20}$$

$$N = \begin{pmatrix} N_{11} \\ N_{22} \\ \sqrt{2}N_{12} \end{pmatrix}, \qquad M = \begin{pmatrix} M_{11} \\ M_{22} \\ \sqrt{2}M_{12} \end{pmatrix}, \qquad m = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}, \tag{21}$$

$$C = \begin{pmatrix} C_{1111} & C_{1122} & \sqrt{2}C_{1112} \\ C_{1122} & C_{2222} & \sqrt{2}C_{2212} \\ \sqrt{2}C_{1112} & \sqrt{2}C_{2212} & 2C_{1212} \end{pmatrix}, \qquad D = \begin{pmatrix} D_{11} & D_{12} \\ D_{12} & D_{22} \end{pmatrix}. \tag{22}$$

The resulting vector equation resembling Hooke's law then reads as

$$\begin{aligned}
N(x) &= t\,C(x)\gamma(u(x))\,,\\
M(x) &= \frac{t^3}{12}\,C(x)\chi(u(x),\theta(x))\,,\\
m(x) &= t\,k\,D(x)\zeta(u(x),\theta(x))
\end{aligned} \tag{23}$$

and leads for a given force resultant density $f \in \left[L^2(\omega)\right]^3$, a given traction resultant density $g_u \in \left[L^2(\partial\omega_1)\right]^3$ and a given moment resultant density $g_\theta \in \left[L^2(\partial\omega_1)\right]^2$ to the following formula for the potential energy $\Pi(u,\theta)$ of a Naghdi shell

$$\Pi(u,\theta) = \frac{1}{2}\int_\omega t\gamma^\top C\gamma + \frac{t^3}{12}\chi^\top C\chi + tk\zeta^\top D\zeta \, d\mathcal{S} - \int_\omega tf^\top u \, d\mathcal{S} - \int_{\partial\omega_1} {g_u}^\top u + {g_\theta}^\top \theta \, dl \,. \tag{24}$$

The condition for static equilibrium of a Naghdi shell is then given by minimization of the potential energy

$$\min_{(u,\theta)\in\mathcal{U}} \Pi(u,\theta) \,. \tag{25}$$

An often needed special case of Naghdi shells appears for planar midsurfaces. Thus the curvature is zero everywhere on the midsurface ω leading to vanishing second and third fundamental forms

$$b_{\alpha\beta} = 0 \quad , \quad c_{\alpha\beta} = 0 \tag{26}$$

and a constant normal vector a_3. The resulting model is known as the Reissner-Mindlin plate model with the following simplified formulas for the strains

$$\begin{aligned} \gamma_{\alpha\beta}(u_1,u_2) &= \frac{1}{2}\left(u_{\alpha|\beta} + u_{\beta|\alpha}\right) \,, \\ \chi_{\alpha\beta}(\theta) &= \frac{1}{2}\left(\theta_{\alpha|\beta} + \theta_{\beta|\alpha}\right) \,, \\ \zeta_\alpha(u_3,\theta) &= \frac{1}{2}\left(\theta_\alpha + u_{3,\alpha}\right) \,. \end{aligned} \tag{27}$$

While the condition for an equilibrium state of the plate remains the same as in the general case of shells

$$\begin{aligned} \min_{(u,\theta)\in\mathcal{U}} \Pi(u,\theta) &= \frac{1}{2}\int_\omega t\gamma^\top(u_1,u_2)C\gamma(u_1,u_2) + \frac{t^3}{12}\chi^\top(\theta)C\chi(\theta) \\ &\quad + tK\zeta^\top(u_3,\theta)D\zeta(u_3,\theta)\, d\mathcal{S} - \int_\omega tf^\top u \, d\mathcal{S} - \int_{\partial\omega_1} {g_u}^\top u + {g_\theta}^\top \theta \, dl \,, \end{aligned} \tag{28}$$

this minimization problem can be separated into the membrane problem

$$\min_{(u_1,u_2)\in\mathcal{U}} \frac{1}{2}\int_\omega t\gamma^\top(u_1,u_2)C\gamma(u_1,u_2)\, d\mathcal{S} - \int_\omega t\left(f_1u_1 + f_2u_2\right)\, d\mathcal{S} - \int_{\partial\omega_1} \left(g_{u1}u_1 + g_{u2}u_2\right)\, dl$$

and the so-called Reissner-Mindlin problem

$$\min_{(u_3,\theta)\in\mathcal{U}} \int_\omega \frac{t^3}{24}\chi^\top(\theta)C\chi(\theta) + \frac{t}{2}K\zeta^\top(u_3,\theta)D\zeta(u_3,\theta)\, d\mathcal{S} - \int_\omega tf_3u_3 \, d\mathcal{S} - \int_{\partial\omega_1} g_{u3}u_3 + {g_\theta}^\top\theta \, dl$$

when looking for static equilibrium for constant material. But as we intend to find an optimal material distribution in the next sections, we will only consider the entire minimization problem (28) in the case of plates.

2.3. Admissible Materials and Measures for Stiffness and Weight

After having found a proper description of the elastic behaviour of Naghdi shells our goal is to determine the optimal material distribution which yields the stiffest structure. There are various methods to this end differing in the set of admissible materials. In this article we focus on Free Material Optimization, thus there is no restriction on a specific material symmetry. All materials are allowed hence the full elasticity matrices C and D are used as optimization variables. To admit holes and material-no-material situations the material matrices are chosen from $C \in [L^\infty(\omega)]^{3\times 3}$ and $D \in [L^\infty(\omega)]^{2\times 2}$. As seen in (19) the material matrices are symmetric. Moreover, as they are describing material in a linear elastic theory they have to be positive semidefinite:

$$C = C^\top \succeq 0 \quad , \quad D = D^\top \succeq 0 \,. \tag{29}$$

Thus the set of admissible materials is given by

$$\tilde{\mathcal{C}} := \left\{ (C, D) \in [L^\infty(\omega)]^{3\times 3} \times [L^\infty(\omega)]^{2\times 2} \,\middle|\, C = C^\top \succeq 0 \,,\, D = D^\top \succeq 0 \right\} . \tag{30}$$

After defining the admissible set for the material matrices the next step is to determine a measure for the stiffness of the structure, which we intend to maximize. A commonly used measure is the compliance, which describes how much a structure will deform under the given set of loads f, g_u and g_θ and is defined as

$$\begin{aligned} \operatorname{comp}(C, D) &= \int_\omega t f^\top u \, d\mathcal{S} + \int_{\partial\omega_1} \left(g_u^\top u + g_\theta^\top \theta \right) \, dl \qquad (31) \\ &= \max_{(u,\theta)\in\mathcal{U}} - \int_\omega \left(t\gamma^\top C\gamma + \frac{t^3}{12} \chi^\top C\chi + t\zeta^\top D\zeta \right) \, d\mathcal{S} \\ &\qquad\qquad + 2\int_\omega t f^\top u \, d\mathcal{S} + 2\int_{\partial\omega_1} \left(g_u^\top u + g_\theta^\top \theta \right) \, dl \\ &= \max_{(u,\theta)\in\mathcal{U}} -2\,\Pi_{C,D}(u,\theta) = - \min_{(u,\theta)\in\mathcal{U}} 2\,\Pi_{C,D}(u,\theta) \,. \end{aligned}$$

Thus the compliance is equal to twice the negative potential energy in static equilibrium. As the compliance is a measure on how much a structure will yield to the applied forces a minimization of the compliance with respect to the design variables will lead to the stiffest structure possible.

Finally a measure on the amount of distributed material is required. If there is no limit to the amount of used material the stiffest solution will consist of filling the entire design domain with material leading to unusable and costly designs. Thus a constraint on the used amount of material is necessary. Usually the density of the chosen material is taken, but as the material variable in Free Material is not restricted to material, that is already existent in nature, the optimal material might be entirely new with an unknown density. Therefore a norm on the material matrices C and D is taken as an indicator on how much material was used at a certain spot of the design domain. Throughout this article we will use the summed traces of the matrices C and D and are thus able to measure the amount of used material at a certain spot $x \in \omega$ by

$$t\,\mathrm{tr}(C(x)) + \frac{1}{2}\,t\,\mathrm{tr}(D(x)) \tag{32}$$

and the amount of material used for the entire structure by

$$\mathrm{vol}(C, D) := \int_\omega t \left(\mathrm{tr}(C(x)) + \frac{1}{2}\mathrm{tr}(D(x)) \right) \, d\mathcal{S}\,. \tag{33}$$

The factor $\frac{1}{2}$ is used for consistency with the trace of the three-dimensional elasticity tensor for solid material.

2.4. Optimization Problems

With the above definitions it is possible to set up various optimization problem formulations for the Free Material Optimization problem for shells. On the one hand there is the minimum-compliance formulation already introduced in [12, 13]. Here the compliance is taken as objective function and minimized to obtain the stiffest design for the structure. As the available material resources are limited a volume constraint of the form

$$\int_\omega t \left(\mathrm{tr}(C) + \frac{1}{2}\mathrm{tr}(D) \right) \, d\mathcal{S} \le V \tag{34}$$

is added to the problem. Furthermore arbitrarily high material concentrations at single points are to be avoided. For this purpose we add the following box constraints

$$0 \le \rho^- \le t\,\mathrm{tr}(C(x)) + \frac{1}{2}\,t\,\mathrm{tr}(D(x)) \le \rho^+ \quad \text{a.e. in } \omega\,, \tag{35}$$

which results in the proximate problem formulation

$$\begin{aligned}
\min_{(C,D)\in\tilde{\mathcal{C}}} \max_{(u,\theta)\in\mathcal{U}} J((C,D),(u,\theta)) \quad &:= \quad -\int_\omega \frac{t}{2}\gamma^\top C\gamma + \frac{t^3}{24}\chi^\top C\chi + \frac{t}{2}k\zeta^\top D\zeta \, dS \\
&\qquad + \int_\omega t f^\top u \, dS + \int_{\partial\omega_1} {g_u}^\top u + {g_\theta}^\top \theta \, dl \qquad (P_D) \\
\text{subject to} \quad &\quad \int_\omega t \left(\mathrm{tr}(C) + \frac{1}{2}\mathrm{tr}(D) \right) \, d\mathcal{S} \le V \\
&\quad \rho^- \le t\,\mathrm{tr}(C(x)) + \frac{1}{2}\,t\,\mathrm{tr}(D(x)) \le \rho^+ \text{ a.e. in } \omega
\end{aligned}$$

Problem (P_D) has been studied elaborately in [12] with a focus on existence of solutions and dual problem formulations. Existence of at least one optimal solution is guaranteed by the following tenet.

Theorem 1. *Problem (P_D) has an optimal solution $((C^*, D^*), (u^*, \theta^*)) \in \mathcal{C} \times \mathcal{U}$.*

Proof. The proof is given in [12] and follows ideas from [24]. □

Moreover, it is possible to prove that problem (P_D) is the dual of a infinite-dimensional nonlinear convex semi-definite program (SDP) according to the next statement.

Theorem 2. *Problem (P_D) is equivalent to the Lagrange dual problem of*

$$\begin{aligned}
\max_{\substack{(u,\theta)\in\mathcal{U}\\ \alpha\in\mathbb{R}_+\\ \beta_{u,l}\in L^1(\omega)\\ \beta_{u,l}\geq 0}} \quad & \int_\omega t f^\top u \, d\mathcal{S} + \int_{\partial\omega_1} ({g_u}^\top u + {g_\theta}^\top \theta) dl - \alpha V - \rho^+ \int_\omega \beta_u d\mathcal{S} + \rho^- \int_\omega \beta_l d\mathcal{S} \\
\text{subject to} \quad & \frac{t}{2}\gamma(u)\gamma(u)^\top + \frac{t^3}{24}\chi(u,\theta)\chi(u,\theta)^\top - t(\alpha + \beta_u - \beta_l)\mathbb{1}_3 \preceq 0 \\
& \frac{t}{2}k\zeta(u,\theta)\zeta(u,\theta)^\top - \frac{t}{2}(\alpha + \beta_u - \beta_l)\mathbb{1}_2 \preceq 0
\end{aligned} \qquad (P_P)$$

where $\mathbb{1}_n$ denotes the unit matrix in $\mathbb{R}^n$.

Proof. The proof is also given in [12] and uses results from [37, 5]. □

The obtained problem formulation (P_P) is preferred over the saddle-point problem (P_D) as the material matrices – whose discrete counterparts highly increase the number of optimization variables in the optimization problem – are hidden in the problem formulation as Lagrange multipliers of the matrix inequality constraints. Even more importantly, problem (P_P) is a convex optimization problem allowing us to find global optima by checking local optimality conditions.

On the other hand there are problem formulations that minimize the weight under a compliance constraint, which also provide stiff and light-weight structures. The material measure introduced in (33) is taken as objective function to find the structure with minimal weight. To ensure that the structure's stiffness is sufficient to carry the applied loads a compliance constraint of the form

$$\int_\omega t f^\top u \, d\mathcal{S} + \int_{\partial\omega_1} {g_u}^\top u + {g_\theta}^\top \theta \, dl \leq c \qquad (36)$$

is added to the optimization problem. As in (P_D) and (P_P) upper and lower bounds on the material tensors are introduced. While the upper bound is again a bound on the trace of the tensors C and D, the lower bound is a bound on the eigenvalues. If the parameter ε is positive, all eigenvalues also have to be positive resulting in positive definite material matrices. Thus no holes can be created in the structure in contrast to the previous problem formulations. The resulting problem formulation known as the minimum-weight formulation of the Free Material Optimization problem for shells is then a nonconvex nonlinear SDP.

$$\begin{aligned}
\min_{\substack{(u,\theta)\in\mathcal{U}\\(C,D)\in\tilde{\mathcal{C}}}} \int_\omega t\cdot \mathrm{tr}C + \frac{t}{2}\cdot \mathrm{tr}D\, dS & \\
\text{subject to} \qquad C \succeq 0\,;\ D &\succeq 0 \\
\begin{pmatrix} tC & 0 \\ 0 & \frac{t}{2}D \end{pmatrix} - \varepsilon \mathbb{1}_5 &\succeq 0 \\
t\cdot \mathrm{tr}C + \frac{t}{2}\cdot \mathrm{tr}D &\leq \rho^+ \\
\int_\omega t f^\top u\, d\mathcal{S} + \int_{\partial\omega_1} {g_u}^\top u + {g_\theta}^\top \theta\, dl &\leq c \\
\int_\omega t\gamma^\top(u)C\gamma(v) + \frac{t^3}{12}\chi^\top(u,\theta)C\chi(v,\eta) + tk\zeta^\top(u,\theta)D\zeta(v,\eta)\, d\mathcal{S} &= \\
= \int_\omega t f^\top v\, d\mathcal{S} + \int_{\partial\omega_1} {g_u}^\top v + {g_\theta}^\top \eta\, dl \quad \forall(v,\eta)\in\mathcal{U} &
\end{aligned} \tag{37}$$

Remark: The given problem formulations remain also valid for Reissner-Mindlin plates. As the material tensor C is chosen as optimization variable for the Free Material Optimization problem and appears both in the membrane and the bending term of the strain energy the uncoupling into the membrane and the Reissner-Mindlin problem as known from the equilibrium problem is not possible anymore. Thus all given problem formulations are also applicable for plates, only the formulas for the strains are simplified according to (27). Also note that the minimum-compliance formulation for Free Material Optimization for Reissner-Mindlin plates was already given in [2], where a solution is obtained by analytic derivation of the optimal material properties.

3. Discrete Problem Formulation

3.1. Finite Elements for Shells

As we are interested in a numerical solution of the optimization problems introduced in the last section, we employ a finite element method to obtain discrete variants for the appearing quantities [7]. The midsurface ω is split up into M quadrangular elements ω_m with a total number of n element nodes for the entire mesh. The material matrices $C(x)$ and $D(x)$ are assumed to be elementwise constant, thus they are approximated by vectors containing one element matrix per element such as $(C_1,\ldots,C_M)$ and $(D_1,\ldots,D_M)$, where $C_m\in\mathbb{R}^{3\times3}$ and $D_m\in\mathbb{R}^{2\times2}$ for all $m=1,\ldots,M$. A bilinear approximation using the bilinear 2D Lagrange shape functions $\vartheta_i(r,s)$ is taken for the displacements

$$U^{3D} = \sum_{i=1}^{n} \vartheta_i(r,s)\left(u^{(i)} + z\frac{t}{2}\theta^{(i)}\right)\,. \tag{38}$$

Hence it is guaranteed that the material fibres remain straight and unstretched during deformation, thus the Reissner-Mindlin assumption is not violated. From (38) follow the strain-displacement matrices B_i^γ, B_i^χ and B_i^ζ for the membrane strains γ, the bending strains χ and the shear strains ζ, respectively, as

$$B_i^\gamma = \begin{pmatrix} \vartheta_{i|1} & 0 & -b_{11}\vartheta_i & 0 & 0 \\ 0 & \vartheta_{i|2} & -b_{22}\vartheta_i & 0 & 0 \\ \frac{1}{\sqrt{2}}\vartheta_{i|2} & \frac{1}{\sqrt{2}}\vartheta_{i|1} & -\sqrt{2}\, b_{12}\vartheta_i & 0 & 0 \end{pmatrix}, \tag{39}$$

$$B_i^\chi = \left(\begin{matrix} -b_1^1\vartheta_{i|1} & -b_1^2\vartheta_{i|1} \\ -b_2^1\vartheta_{i|2} & -b_2^2\vartheta_{i|2} \\ -\frac{1}{\sqrt{2}}(b_2^1\vartheta_{i|1} + b_1^1\vartheta_{i|2}) & -\frac{1}{\sqrt{2}}(b_2^2\vartheta_{i|1} + b_1^2\vartheta_{i|2}) \end{matrix} \right. \quad \dots$$

$$\dots \quad \left. \begin{matrix} c_{11}\vartheta_i & \vartheta_{i|1} & 0 \\ c_{22}\vartheta_i & 0 & \vartheta_{i|2} \\ \sqrt{2}c_{12}\vartheta_i & \frac{1}{\sqrt{2}}\vartheta_{i|2} & \frac{1}{\sqrt{2}}\vartheta_{i|1} \end{matrix} \right). \tag{40}$$

$$B_i^\zeta = \begin{pmatrix} \frac{1}{2}b_1^1\vartheta_i & \frac{1}{2}b_1^2\vartheta_i & \frac{1}{2}\vartheta_{i,1} & \frac{1}{2}\vartheta_i & 0 \\ \frac{1}{2}b_2^1\vartheta_i & \frac{1}{2}b_2^2\vartheta_i & \frac{1}{2}\vartheta_{i,2} & 0 & \frac{1}{2}\vartheta_i \end{pmatrix}, \tag{41}$$

Note that due to consistency with the vector-matrix-notation introduced in (20) the factor $\sqrt{2}$ has to be included in the formulas for B_i^γ and B_i^χ. Let now K be the index set of nodes associated with the m-th element ω_m. Then the discrete version of the dyadic products $\gamma\gamma^\top$, $\chi\chi^\top$ and $\zeta\zeta^\top$ are given by

$$A_m^\gamma(u) = \sum_{i,j\in K} \int_{\omega_m} B_j^\gamma U U^\top (B_i^\gamma)^\top dx\,, \tag{42}$$

$$A_m^\chi(u) = \sum_{i,j\in K} \int_{\omega_m} B_j^\chi U U^\top (B_i^\chi)^\top dx\,, \tag{43}$$

$$A_m^\zeta(u) = \sum_{i,j\in K} \int_{\omega_m} B_j^\zeta U U^\top (B_i^\zeta)^\top dx\,. \tag{44}$$

Furthermore the element stiffness matrices of the element ω_m are related to the strain-displacement-matrices by

$$\begin{aligned} K^\gamma(C_m) &= \sum_{i,j\in K} \int_{\omega_m} t_m (B_i^\gamma)^\top C_m B_i^\gamma\, dx\,, \\ K^\chi(C_m) &= \sum_{i,j\in K} \int_{\omega_m} \frac{{t_m}^3}{12} (B_i^\chi)^\top C_m B_i^\chi\, dx\,, \\ K^\zeta(D_m) &= \sum_{i,j\in K} \int_{\omega_m} t_m k (B_i^\zeta)^\top D_m B_i^\zeta\, dx\,. \end{aligned} \tag{45}$$

We also define for further use

$$K_{\text{shell}}(C,D) := \sum_{m=1}^{\text{EltNr}} K^\gamma(C_m) + K^\chi(C_m) + K^\zeta(D_m)\,. \tag{46}$$

3.2. Discretized Optimization Problems

It is now possible to state explicitly the discretized versions of the optimization problems derived in section 2.4. Assume that all quantities – forces, moments and so forth – are substituted by their discrete counterparts. The discretized version of the dual Free Material Optimization problem for shells in the minimum-compliance formulation (P_P) can be determined as

$$\begin{aligned}
&\max_{\substack{(u,\theta)\in\mathcal{U}\\ \alpha\in\mathbb{R}_+\\ \beta_u,\beta_l\in{\mathbb{R}_+}^M}} && \sum_{i=1}^{n}\left(tf_iu_i-\rho^+\beta_{u\,i}+\rho^-\beta_{l\,i}\right)+\sum_{i\in\partial\omega_1}(g_{u\,i}u_i+g_{\theta\,i}\theta_i)dl-V\alpha \\
&\text{subject to} && \frac{t}{2}A_m^{\gamma}(u)+\frac{t^3}{24}A_m^{\chi}(u,\theta)-t(\alpha+\beta_u-\beta_l)\mathbb{1}_3\preceq 0 \qquad (47)\\
& && \frac{t}{2}KA_m^{\zeta}(u,\theta)-\frac{t}{2}(\alpha+\beta_u-\beta_l)\mathbb{1}_2\preceq 0\,.
\end{aligned}$$

It can be seen that (47) is a finite-dimensional convex nonlinear semidefinite program. We now proceed to the minimum weight formulation as given in (37). The discretized version of this problem is given by

$$\min_{\substack{(u,\theta)\in\mathcal{U}_h\\ (C,D)\in\mathcal{C}_h}} \sum_{m=1}^{\text{EltNr}} t_m\cdot\text{tr}C_m+\frac{t_m}{2}\cdot\text{tr}D_m \qquad (48)$$

$$\begin{aligned}
\text{subject to}\qquad C_m\succeq 0;\ D_m &\succeq 0\quad \forall\, m=1,\dots,\text{EltNr}\\
\begin{pmatrix} t_mC_m & 0\\ 0 & \frac{t_m}{2}D_m\end{pmatrix}-\varepsilon_m\mathbb{1}_5 &\succeq 0\quad \forall\, m=1,\dots,\text{EltNr}\\
t_m\cdot\text{tr}C_m+\frac{t_m}{2}\cdot\text{tr}D_m &\leq \rho^+\ \ \forall\, m=1,\dots,\text{EltNr}\\
\left(\sum_{m=1}^{\text{EltNr}}K^{\gamma}(C_m)+K^{\chi}(C_m)+K^{\zeta}(D_m)\right)(u,\theta) &= tf+g_u+g_\theta=f_h\\
{f_h}^{\top}(u,\theta) &\leq c
\end{aligned}$$

thereby yielding a nonlinear and nonconvex SDP. Due to the positive definiteness of C_m and D_m for all $m=1,\dots,\text{EltNr}$ it follows that the element stiffness matrices as defined in (45) are again positive definite which is also true for $K_{\text{shell}}(C,D)$. Hence the equilibrium condition can be used to substitute the displacements (u,θ) by

$$\begin{aligned}
\left(\sum_{m=1}^{\text{EltNr}}K^{\gamma}(C_m)+K^{\chi}(C_m)+K^{\zeta}(D_m)\right)(u,\theta) &= K_{\text{shell}}(C,D)(u,\theta)=f_h\\
\Rightarrow\quad (u,\theta) &= K_{\text{shell}}(C,D)^{-1}f_h \qquad (49)
\end{aligned}$$

thus removing the state variable (u, θ) from the optimization problem and leading to the reduced minimal weight-formulation

$$\begin{aligned} \min_{(C,D)\in\mathcal{C}_h} \sum_{m=1}^{\mathrm{EltNr}} t_m \cdot \mathrm{tr} C_m + \frac{t_m}{2} \cdot \mathrm{tr} D_m & & & \qquad (50) \\ \text{subject to} \qquad C_m \succeq 0\,;\ D_m &\succeq 0 & & \forall\, m = 1,\dots,\mathrm{EltNr} \\ \begin{pmatrix} t_m C_m & 0 \\ 0 & \frac{t_m}{2} D_m \end{pmatrix} - \varepsilon_m \mathbb{1}_5 &\succeq 0 & & \forall\, m = 1,\dots,\mathrm{EltNr} \\ t_m \cdot \mathrm{tr} C_m + \frac{t_m}{2} \cdot \mathrm{tr} D_m &\le \rho^+ & & \forall\, m = 1,\dots,\mathrm{EltNr} \\ {f_h}^\top K_{\mathrm{shell}}(C,D)^{-1} f_h &\le c & & \end{aligned}$$

This step restores convexity of the problem, therefore the Free Material Optimization problem for shells in the reduced minimum weight-formulation is a convex nonlinear SDP problem.

4. Free Material Optimization for Shells with Displacement Constraints

In the previous sections we have dealt with the problem of finding the stiffest structure possible for a given set of loads using only a limited amount of material. While the ability of a structure to sustain the applied loads is vital, numerous other requirements have to be met when facing real world problems. Although some desired features – for example aesthetic appearance – can not be captured by mathematical formulas, it is possible to treat several additional optimization constraints in the context of problem (50).

We first approach the problem of adding displacement constraints to the original problem formulation. Constraints on the displacements can be used in manifold ways:

- to prescribe a shape of the structure deformed under the loads. For example a certain shape of an airplane's fuselage might enhance its aerodynamic features or a part of a designed structure might be used as a reflector or an antenna demanding a distinct shape.
- to ensure movement in a certain direction for some parts of the structure. This is usually the case when building mechanisms like grippers and force inverters.
- to appoint minimal or fixed displacements in certain areas of the design domain or for certain nodes of the finite element mesh.

As in [19] we will treat displacement constraints solely in a discrete context. We introduce these constraints as linear constraints of the form

$$C_{\mathrm{DC}}(u, \theta) \le d_{\mathrm{DC}} \qquad (51)$$

where $C_{\mathrm{DC}} \in \mathbb{R}^{r\times n}$ and $d \in \mathbb{R}^r$ are given. Thus the primal Free Material Optimization problem for shells including displacement constraints is given by

$$\begin{aligned} \min_{\substack{(u,\theta)\in\mathcal{U}_h\\(C,D)\in\mathcal{C}_h}} \sum_{m=1}^{\mathrm{EltNr}} t_m \cdot \mathrm{tr}C_m + \frac{t_m}{2}\cdot \mathrm{tr}D_m & & & \qquad (52)\\ \text{subject to} \qquad C_m \succeq 0;\ D_m &\succeq& 0 & \quad \forall\, m = 1,\dots,\mathrm{EltNr}\\ \begin{pmatrix} t_m C_m & 0\\ 0 & \frac{t_m}{2} D_m\end{pmatrix} - \varepsilon_m \mathbb{1}_5 &\succeq& 0 & \quad \forall\, m = 1,\dots,\mathrm{EltNr}\\ t_m \cdot \mathrm{tr}C_m + \frac{t_m}{2}\cdot \mathrm{tr}D_m &\le& \rho^+ & \quad \forall\, m = 1,\dots,\mathrm{EltNr}\\ \left(\sum_{m=1}^{\mathrm{EltNr}} K^\gamma(C_m) + K^\chi(C_m) + K^\zeta(D_m)\right)(u,\theta) &=& f_h & \\ {f_h}^\top(u,\theta) &\le& c & \\ C_{\mathrm{DC}}(u,\theta) &\le& d_{\mathrm{DC}} & \end{aligned}$$

Apparently the Free Material Optimization problem for shells including displacement constraints is again a nonlinear semidefinite program. Yet there is a crucial difference to the previous problem (50) concerning existence of solutions. The introduced displacement constraints modify the admissible set $\mathcal{U}_h$ and might remove all solutions (u,θ) able to carry the structure (and thereby fulfilling the compliance constraint). Thus the set of admissible displacements fulfilling both the compliance and the displacement constraints might be empty. In this case there exists no optimal solution to problem (52). Thus existence of solutions can not be shown for the Free Material Optimization problem for shells with displacement constraints.

Furthermore we would like to emphasize the difference between the displacement constraints used here and contact conditions (as found for example in [1]). As explained in [14] we obtain unilateral contact conditions by adding linear displacement constraints to the minimum-potential-energy formulation (P_D). The difference between the two problems becomes apparent when the structure shares a common interface with an obstacle inside or at the boundary of the design domain. While the normal stresses at the interfaces vanish in the case of displacement constraints this is not necessary for unilateral contact conditions. Here the reaction forces at the interface do not have to be equal to zero as long as the normal stresses fulfill the static equilibrium equations. Thus displacement constraints are a special case of unilateral contact conditions.

5. Free Material Optimization for Shells with Stress and Strain Constraints

The next problem to deal with is the appearance of high stresses in the optimal structure [10]. High stresses are critical as when reaching the yield stress of a material it will deform according to a plastic model. Thus the used elastic model is not sufficient to describe the behaviour of the structure. When increasing the stresses even more structural failure might occur as the material rips apart.

We will treat stress constraints analogous to [18, 19]. Hence the norms of the stresses integrated over an element ω_m are bounded from above in the discretized problem formulation. These constraints are normalized by the measure of the element $|\omega_m|$ and the upper bound on the trace of the material ρ^+.

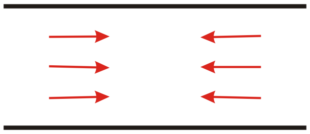

Fig. 2: Membrane stress

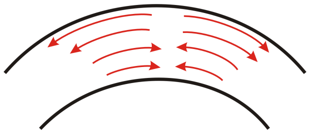

Fig. 3: Bending stress

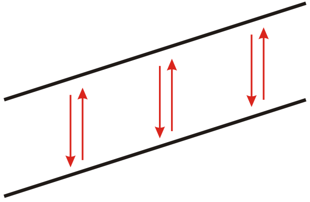

Fig. 4: Shear stress

In the case of shells appear three different stresses: membrane, bending and shear stresses. This gives rise to the question on whether to combine the constraints for these stresses or to treat them in separate constraints. To answer this a look at deformations causing the different stresses is helpful. In Fig. 2 a compression in the midplane causing membrane stresses is shown, while in Fig. 3 a typical bending deformation is depicted. Although the resulting stresses are different, their directions stay in the plane given by the midsurface. Therefore we consider these stresses as in-plane stresses. One can clearly see when looking at the shear stresses in Fig. 4, that they are out-of-plane stresses. Thus we will add a combined constraint for the in-plane stresses bounded by s_σ^{ip} and a single constraint for the out-of-plane stresses bounded by s_σ^{oop} to the optimization problem (48) resulting in the following Free Material Optimization problem for shells with stress constraints in the minimum weight-formulation.

$$
\begin{aligned}
\min_{\substack{(u,\theta)\in\mathcal{U}_h\\(C,D)\in\mathcal{C}_h}} \sum_{m=1}^{\mathrm{EltNr}} t_m\cdot\mathrm{tr}C_m+\frac{t_m}{2}\cdot\mathrm{tr}D_m \qquad\qquad\qquad\qquad & \qquad (53)\\
\text{subject to} \qquad C_m\succeq 0\,;\; D_m \;&\succeq\; 0 \quad \forall m=1,\dots,\mathrm{EltNr}\\
\begin{pmatrix} t_mC_m & 0\\ 0 & \frac{t_m}{2}D_m\end{pmatrix}-\varepsilon_m\mathbb{1}_5 \;&\succeq\; 0 \quad \forall m=1,\dots,\mathrm{EltNr}\\
t_m\cdot\mathrm{tr}C_m+\frac{t_m}{2}\cdot\mathrm{tr}D_m \;&\leq\; \rho^+ \;\; \forall m=1,\dots,\mathrm{EltNr}\\
\sum_{g=1}^{\mathrm{GpNr}} \|CB^{\gamma}_{m,g}(u,\theta)\|^2+\|CB^{\chi}_{m,g}(u,\theta)\|^2 \;&\leq\; s_\sigma^{\mathrm{ip}}(\rho^+)^2|\omega_m| \quad \forall m=1,\dots,\mathrm{EltNr}\\
\sum_{g=1}^{\mathrm{GpNr}} \|DB^{\zeta}_{m,g}(u,\theta)\|^2 \;&\leq\; s_\sigma^{\mathrm{oop}}(\rho^+)^2|\omega_m| \quad \forall m=1,\dots,\mathrm{EltNr}\\
\left(\sum_{m=1}^{\mathrm{EltNr}} K^{\gamma}(C_m)+K^{\chi}(C_m)+K^{\zeta}(D_m)\right)(u,\theta) \;&=\; f_h\\
{f_h}^{\top}(u,\theta) \;&\leq\; c
\end{aligned}
$$

It is also possible to include constraints on the strains instead of the stresses. This might be helpful in the case of low material densities, which allow large deformations and strains without violating the stress constraints. Analogous to the stress constrained problem (53) the integral over the norms of the strains are bounded from above in the discretized problem. Again we differ between in-plane and out-of-plane behaviour and obtain the Free Material Optimization problem for shells with strain constraints in the minimum weight-formulation.

$$
\begin{aligned}
\min_{\substack{(u,\theta)\in\mathcal{U}_h\\(C,D)\in\mathcal{C}_h}} \sum_{m=1}^{\mathrm{EltNr}} t_m \cdot \mathrm{tr} C_m + \frac{t_m}{2} \cdot \mathrm{tr} D_m & & \\
\text{subject to} \qquad C_m \succeq 0\,;\; D_m &\succeq 0 \quad \forall\, m = 1,\dots,\mathrm{EltNr} \\
\begin{pmatrix} t_m C_m & 0 \\ 0 & \frac{t_m}{2} D_m \end{pmatrix} - \varepsilon_m \mathbb{1}_5 &\succeq 0 \quad \forall\, m = 1,\dots,\mathrm{EltNr} \\
t_m \cdot \mathrm{tr} C_m + \frac{t_m}{2} \cdot \mathrm{tr} D_m &\leq \rho^+ \;\; \forall\, m = 1,\dots,\mathrm{EltNr} \\
\sum_{g=1}^{\mathrm{GpNr}} \| B^{\gamma}_{m,g}(u,\theta) \|^2 + \| B^{\chi}_{m,g}(u,\theta) \|^2 &\leq s_e^{\mathrm{ip}} (\rho^+)^2 |\omega_m| \quad \forall\, m = 1,\dots,\mathrm{EltNr} \\
\sum_{g=1}^{\mathrm{GpNr}} \| B^{\zeta}_{m,g}(u,\theta) \|^2 &\leq s_e^{\mathrm{oop}} (\rho^+)^2 |\omega_m| \quad \forall\, m = 1,\dots,\mathrm{EltNr} \\
\left(\sum_{m=1}^{\mathrm{EltNr}} K^{\gamma}(C_m) + K^{\chi}(C_m) + K^{\zeta}(D_m) \right)(u,\theta) &= f_h \\
{f_h}^{\top}(u,\theta) &\leq c
\end{aligned}
\tag{54}
$$

6. Free Material Optimization for Shells with Vibration Constraints

Next we tend to vibration constraints. A structure subjected to periodic forces, whose frequency is nearby the structure's natural frequency, omits resonance behavior. It gains energy from the external excitation and its amplitude will increase until the structure rips apart. One of the most prominent examples of this behavior known as resonance disaster is Angers bridge, a suspension bridge leading across the Maine river in France, who collapsed 1850 under soldiers marching in cadence.

An effective strategy to avoid structural failure due to vibrational resonance, which we will employ in this article, is raising a structure's fundamental eigenfrequency. The eigenfrequencies are determined by solving the dynamic problem in the absence of external loading providing the free vibrations of the structure. Thus we need the dynamic linear elasticity problem for Naghdi shells, but up to now we only worked with the static elasticity problem. This problem formulation was obtained by considering the general three-dimensional static elasticity problem and introducing the approximations of

Naghdi's shell model. Hence it is natural to start now from the general three-dimensional dynamic elasticity problem and use the same approximations. The three-dimensional dynamic elasticity problem is well-known and according to [4] given by

Find a function $\tau \to U(\xi,\tau)$ *of* $[0,T] \mapsto \mathcal{V}$ *such that*

$$\int_{\mathcal{R}} \rho^* \frac{\partial^2 U}{\partial \tau^2} V \, d\mathcal{R} + \int_{\mathcal{R}} E^{*\,ijkl} \gamma^*_{ij}(U) \gamma^*_{kl}(V) \, d\mathcal{R} = 0 \qquad \forall\, V \in \mathcal{V} \tag{55}$$

$$U(\xi,0) = U_0(\xi) \quad , \quad \frac{\partial U}{\partial \tau}(\xi,0) = U_1(\xi)$$

where U denotes the displacement in three dimensions, depending on time τ and the three-dimensional space variable ξ. $\partial\mathcal{R}_0$ denotes the clamped part of the boundary $\partial\mathcal{R}$ of the shell body $\mathcal{R}$, hence the displacements U lie in $\mathcal{V} := \left\{ U \in \left[H^1(\mathcal{R})\right]^3 \,\middle|\, U|_{\partial\mathcal{R}_0} = 0 \right\}$, where it is assumed that $|\partial\mathcal{R}_0| > 0$. Furthermore γ^*_{ij} is the covariant strain tensor and $E^{*\,ijkl}$ is the contravariant elasticity tensor relative to the local three-dimensional basis $\{g_1, g_2, g_3\}$ of the shell and ρ^* is its mass density.

It is now possible to insert the assumptions of Naghdi's shell model into the general three-dimensional problem formulation (55) and integrate over the shell's thickness to obtain two-dimensional dynamic equations for the vibrations of Naghdi shells. The bidimensional approximation for the latter term was already introduced in equation (24) as

$$\int_{\mathcal{R}} E^{*\,ijkl} \gamma^*_{ij}(U) \gamma^*_{kl}(V) \, d\mathcal{R} \;\approx\; a((u,\theta),(v,\eta)) \tag{56}$$

where

$$\begin{aligned} a((u,\theta),(v,\eta)) &= \int_{\omega} t\gamma^{\top}(u) C \gamma(v) + \frac{t^3}{12} \chi^{\top}(u,\theta) C \chi(v,\eta) + tk\zeta^{\top}(u,\theta) D \zeta(v,\eta) \, d\mathcal{S} \\ &= \int_{\omega} tC^{\alpha\beta\lambda\mu} \left[\gamma_{\alpha\beta}(u)\gamma_{\lambda\mu}(v) + \frac{t^2}{12} \chi_{\alpha\beta}(u,\theta)\chi_{\lambda\mu}(v,\eta) \right] \\ &\quad + tkD^{\alpha\lambda} \zeta_\alpha(u,\theta)\zeta_\lambda(v,\eta)\sqrt{a}\, d\xi^1 d\xi^2 \end{aligned} \tag{57}$$

Thus only a bidimensional approximation of the first term has to be found. To this end the approximation for the displacements of Naghdi shells (13) and the definition of the infinitesimal volume $d\mathcal{R}$ as given in (11) are inserted

$$\int_{\mathcal{R}} \rho^* \frac{\partial^2 U}{\partial \tau^2} V \, d\mathcal{R} \;=\; \int_{\mathcal{R}} \rho^* \frac{\partial^2}{\partial \tau^2} \left(u + \xi^3 \theta_\alpha a^\alpha \right) \left(v + \xi^3 \eta_\beta a^\beta \right) \sqrt{g}\, d\xi^1 d\xi^2 d\xi^3$$

Now it is possible to employ the relation between the three-dimensional metric g_{ij} and the first fundamental form $a_{\alpha\beta}$ of the midsurface from equation (12):

$$\begin{aligned}
&\int_{\mathcal{R}} \rho^* \frac{\partial^2 U}{\partial \tau^2} V \, d\mathcal{R} = \\
&= \int_{\mathcal{R}} \rho^* \left(\ddot{u} + \xi^3 \ddot{\theta}_\alpha a^\alpha \right) \left(v + \xi^3 \eta_\beta a^\beta \right) \left(1 - 2\mathfrak{H}\, \xi^3 + \mathfrak{K} \left(\xi^3\right)^2 \right) \sqrt{a}\, d\xi^1 d\xi^2 d\xi^3 \\
&= \int_{\mathcal{R}} \rho^* \left(\ddot{u} v + \xi^3 \ddot{u} \eta_\beta a^\beta + \xi^3 v \ddot{\theta}_\alpha a^\alpha + \left(\xi^3\right)^2 \ddot{\theta}_\alpha \eta_\beta a^{\alpha\beta} \right) \left(1 - 2\mathfrak{H}\, \xi^3 + \mathfrak{K} \left(\xi^3\right)^2 \right) d\xi^3 d\mathcal{S} \\
&= \int_\omega \int_{-\frac{t}{2}}^{\frac{t}{2}} \rho^* \Big(\ddot{u} v + \xi^3 \left(\ddot{u} \eta_\beta a^\beta + v \ddot{\theta}_\alpha a^\alpha \right) + \left(\xi^3\right)^2 \ddot{\theta}_\alpha \eta_\beta a^{\alpha\beta} - 2\xi^3 \mathfrak{H}\, \ddot{u} v \\
&\qquad - 2 \left(\xi^3\right)^2 \mathfrak{H} \left(\ddot{u} \eta_\beta a^\beta + v \ddot{\theta}_\alpha a^\alpha \right) - 2 \left(\xi^3\right)^3 \mathfrak{H}\, \ddot{\theta}_\alpha \eta_\beta a^{\alpha\beta} + \left(\xi^3\right)^2 \mathfrak{K}\, \ddot{u} v \\
&\qquad + \left(\xi^3\right)^3 \mathfrak{K} \left(\ddot{u} \eta_\beta a^\beta + v \ddot{\theta}_\alpha a^\alpha \right) + \left(\xi^3\right)^4 \mathfrak{K}\, \ddot{\theta}_\alpha \eta_\beta a^{\alpha\beta} \Big) \, d\xi^3 d\mathcal{S}
\end{aligned}$$

The next step is integration over the thickness of the shell using the formulas

$$\int_{-\frac{t}{2}}^{\frac{t}{2}} \left(\xi^3\right)^{2n} d\xi^3 = \frac{2}{(2n+1) 2^{2n+1}} \, t^{2n+1} \quad , \quad \int_{-\frac{t}{2}}^{\frac{t}{2}} \left(\xi^3\right)^{2n+1} d\xi^3 = 0$$

under the assumption of a constant material density ρ in the normal direction resulting in

$$\begin{aligned}
\int_{\mathcal{R}} \rho^* \frac{\partial^2 U}{\partial \tau^2} V \, d\mathcal{R} \;\; = \;\; & \int_\omega \rho \Big(t \ddot{u} v + \frac{t^3}{12} \ddot{\theta}_\alpha \eta_\beta a^{\alpha\beta} - 2 \frac{t^3}{12} \mathfrak{H} \left(\ddot{u} \eta_\beta a^\beta + v \ddot{\theta}_\alpha a^\alpha \right) \\
& + \frac{t^3}{12} \mathfrak{K}\, \ddot{u} v + \frac{t^5}{80} \mathfrak{K}\, \ddot{\theta}_\alpha \eta_\beta a^{\alpha\beta} \Big) \, d\mathcal{S}
\end{aligned}$$

As the shell's thickness is considered to be small the last term including t^5 can be neglected and leads together with the differential geometrical formulas

$$\ddot{u} v = a^{\alpha\beta} \ddot{u}_\alpha v_\beta + \ddot{u}_3 v_3 \; , \quad \ddot{u} \eta_\beta \alpha^\beta = a^{\alpha\beta} \ddot{u}_\alpha \eta_\beta \; , \quad v \ddot{\theta}_\alpha a^\alpha = v_\beta a^{\alpha\beta} \ddot{\theta}_\alpha$$

to the following approximation

$$\int_{\mathcal{R}} \rho^* \frac{\partial^2 U}{\partial \tau^2} V \, d\mathcal{R} \;\; \approx \;\; b((u, \theta), (v, \eta))$$

where

$$\begin{aligned}
b((u, \theta), (v, \eta)) \;\; = \;\; & \int_\omega \rho t \left(1 + \mathfrak{K} \frac{t^2}{12} \right) \left(a^{\alpha\beta} \ddot{u}_\alpha v_\beta + \ddot{u}_3 v_3 \right) \\
& + \rho \frac{t^3}{12} a^{\alpha\beta} \left(\ddot{\theta}_\alpha \eta_\beta - 2\mathfrak{H} \left[\ddot{u}_\alpha \eta_\beta + \ddot{\theta}_\alpha v_\beta \right] \right) d\mathcal{S} \qquad (58)
\end{aligned}$$

Thus the bidimensional dynamic elasticity equations for Naghdi shells in the absence of external loading take the form

Find a function $\tau \to \big(u(\xi^1,\xi^2,\tau),\theta(\xi^1,\xi^2,\tau)\big)$ *of* $[0,T]$ *in* $\mathcal{U}$ *such that*

$$a((u,\theta),(v,\eta)) + b((u,\theta),(v,\eta)) = 0 \quad \forall\,(v,\eta)\in\mathcal{U} \tag{59}$$

where $\mathcal{U}$ is defined as in (14). The solutions to this problem, which describes vibrations of the shell without any effects of external loads and under kinematic boundary conditions, which are time-independent, are called free vibrations. According to [4, 32] they may be expressed as

$$\Big(u(\xi^1,\xi^2,\tau),\theta(\xi^1,\xi^2,\tau)\Big) = (\alpha\cos\Omega t + \beta\sin\Omega t)\,\Big(\tilde u(\xi^1,\xi^2),\tilde\theta(\xi^1,\xi^2)\Big)\,. \tag{60}$$

Here Ω is the vibration frequency. Inserting (60) into (59) and substituting $\tilde u$ and $\tilde\theta$ by u and θ for simplicity yields the following problem

Find triples $(\Omega,u,\theta)\in\mathbb{R}_{++}\times\mathcal{U}$ *such that*

$$a((u,\theta),(v,\eta)) - \Omega^2\,\tilde b((u,\theta),(v,\eta)) = 0 \quad \forall\,(v,\eta)\in\mathcal{U} \tag{61}$$

where

$$\begin{aligned}\tilde b((u,\theta),(v,\eta)) \;&=\; \int_\omega \rho t\left(1+\mathfrak{K}\frac{t^2}{12}\right)\Big(a^{\alpha\beta}u_\alpha v_\beta + u_3 v_3\Big)\\ &\qquad + \rho\frac{t^3}{12}\,a^{\alpha\beta}\left(\theta_\alpha\eta_\beta - 2\mathfrak{H}\left[u_\alpha\eta_\beta+\theta_\alpha v_\beta\right]\right)d\mathcal{S}\end{aligned} \tag{62}$$

Defining $\lambda = \Omega^2$ the vibration problem for shells has been transformed into a time-independent generalized eigenvalue problem:

Find $(\lambda,u,\theta)\in\mathbb{R}_{++}\times\mathcal{U}$ *such that*

$$a((u,\theta),(v,\eta)) = \lambda\,\tilde b((u,\theta),(v,\eta)) \quad \forall\,(v,\eta)\in\mathcal{U} \tag{63}$$

As shown in [36] the solutions to this problem can be used to increase the stability of a structure with respect to vibration phenomena. This is done by raising the structure's fundamental eigenfrequency or equivalently the smallest well defined eigenvalue. Thus the frequency range for resonance behaviour is also shifted and removed from frequency ranges of external excitations. To this end let the smallest well defined eigenvalue $\lambda_{\min}(C,D)$ of problem (63) for material matrices (C,D) from $\tilde{\mathcal{C}}$ be

$$\lambda_{\min}(C,D) := \min\Big\{\lambda \,\Big|\, \exists\,(u,\theta)\in\mathcal{U}\,:\,(\lambda,u,\theta)\text{ solves (63) and }(u,\theta)\notin\ker(\tilde b)\Big\} \tag{64}$$

where

$$\ker(\tilde b) = \Big\{(w,\nu)\,\Big|\,\tilde b((w,\nu),(v,\eta)) = 0 \quad \forall\,(v,\eta)\in\mathcal{U}\Big\}\,.$$

To raise the structure's fundamental eigenfrequency the constraint

$$\lambda_{\min}(C,D) \ge \hat\lambda \tag{65}$$

is added to the Free Material Optimization problem for shells, where $\hat\lambda$ is a prescribed positive lower bound. According to [4] the following statement holds true:

Theorem 3. *Assume that $\Phi \in \left[C^2(\omega)\right]^3$ and that all points on the midsurface $\mathcal{S} = \Phi(\omega)$ are smooth. Then the eigenvalues of problem (63) form an increasing sequence $0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \leq \ldots$ tending to $+\infty$, each of the eigenvalues having a finite multiplicity. Furthermore, there exists an orthonormal basis composed of the eigenvectors associated with the eigenvalues λ_j satisfying*

$$a((u_j,\theta_j),(v,\eta)) = \lambda_j\, \tilde{b}((u_j,\theta_j),(v,\eta)) \quad \forall\,(v,\eta)\in\mathcal{U}$$

and $\tilde{b}((u_i,\theta_i),(u_j,\theta_j)) = \delta_{ij}$.

Together with Corollary A.3 from [36] it is possible to equivalently reformulate the eigenvalue constraint (65) as

$$\inf_{\substack{(v,\eta)\in\mathcal{U}\\ \|(v,\eta)\|=1}} a((v,\eta),(v,\eta)) - \hat{\lambda}\tilde{b}((v,\eta),(v,\eta)) \geq 0 \tag{66}$$

Thus the minimum compliance Free Material Optimization problem for shells (P_D) with vibration constraints reads as

$$\begin{aligned}
\min_{(C,D)\in\tilde{\mathcal{C}}}\ \max_{(u,\theta)\in\mathcal{U}} \quad & -\frac{1}{2}\int_\omega t\gamma^\top C\gamma + \frac{t^3}{12}\chi^\top C\chi + tK\zeta^\top D\zeta\, d\mathcal{S} \\
& \qquad\qquad + \int_\omega tf^\top u\, d\mathcal{S} + \int_{\partial\omega_1} {g_u}^\top u + {g_\theta}^\top\theta\, dl \\
\text{such that} \quad & \int_\omega t\cdot \mathrm{tr}C + \frac{1}{2}t\cdot\mathrm{tr}D\, d\mathcal{S} \leq V \\
& \rho^- \leq t\cdot\mathrm{tr}C + \frac{1}{2}t\cdot\mathrm{tr}D \leq \rho^+ \\
& \inf_{\substack{(v,\eta)\in\mathcal{U}\\ \|(v,\eta)\|=1}} a((v,\eta),(v,\eta)) - \hat{\lambda}\tilde{b}((v,\eta),(v,\eta)) \geq 0
\end{aligned} \tag{67}$$

It is also possible to add the vibration constraints to the minimum weight formulation (37) resulting in

$$\begin{aligned}
&\min_{\substack{(u,\theta)\in\mathcal{U}\\ (C,D)\in\tilde{\mathcal{C}}}} \int_\omega t\cdot\mathrm{tr}C + \frac{t}{2}\cdot\mathrm{tr}D\, dS \\
&\text{subject to} \qquad\qquad C\succeq 0;\ D \ \succeq\ 0 \\
&\qquad \begin{pmatrix} tC & 0 \\ 0 & \frac{t}{2}D \end{pmatrix} - \varepsilon\mathbb{1}_5 \ \succeq\ 0 \\
&\qquad t\cdot\mathrm{tr}C + \frac{t}{2}\cdot\mathrm{tr}D \ \leq\ \rho^+ \\
&\int_\omega tf^\top u\, dS + \int_{\partial\omega_1} {g_u}^\top u + {g_\theta}^\top\theta\, dl \ \leq\ c \\
&\inf_{\substack{(v,\eta)\in\mathcal{U}\\ \|(v,\eta)\|=1}} a((v,\eta),(v,\eta)) - \hat{\lambda}\tilde{b}((v,\eta),(v,\eta)) \ \geq\ 0 \\
&\int_\omega t\gamma^\top(u)C\gamma(v) + \frac{t^3}{12}\chi^\top(u,\theta)C\chi(v,\eta) + tk\zeta^\top(u,\theta)D\zeta(v,\eta)\, d\mathcal{S} = \\
&\qquad = \int_\omega tf^\top v\, d\mathcal{S} + \int_{\partial\omega_1} {g_u}^\top v + {g_\theta}^\top\eta\, dl \quad \forall(v,\eta)\in\mathcal{U}
\end{aligned} \tag{68}$$

To solve problem (68) numerically a discrete version of the vibration constraint has to be added to the discrete problem formulation (48). Thus a discrete formulation for the term $\tilde{b}((u,\theta),(u,\theta))$ is required. To this end a consistent mass matrix as in [11, 22] is introduced. Thus define a matrix $\hat{M}$ entirely filled with blocks of the type

$$\begin{pmatrix} c_1 a^{11} & c_1 a^{12} & 0 & c_2 a^{11} & c_2 a^{12} \\ c_1 a^{21} & c_1 a^{22} & 0 & c_2 a^{21} & c_2 a^{22} \\ 0 & 0 & c_1 & 0 & 0 \\ c_2 a^{11} & c_2 a^{12} & 0 & c_3 a^{11} & c_3 a^{12} \\ c_2 a^{21} & c_2 a^{22} & 0 & c_3 a^{21} & c_3 a^{22} \end{pmatrix}$$

where the abbreviations $c_1 := 1 + \mathfrak{K}\frac{t^2}{12}$, $c_2 := -2\mathfrak{H}\frac{t^2}{12}$ and $c_3 := \frac{t^2}{12}$ are used. Together with vectors $V_{m,g} \in \mathbb{R}^N$, $m = 1, 2, \ldots, \text{EltNr}$, $g = 1, 2, \ldots, \text{GpNr}$, with $\vartheta_j(x_m^g)$, $j \in \mathcal{D}_m$ at the j-th position and zeros otherwise, the mass matrix for Naghdi shells can be defined as

$$\begin{aligned} M(C,D) &= \sum_{m=1}^{\text{EltNr}} M_m(C,D)\ , \\ M_m(C,D) &= \left(t_m \cdot \text{tr} C_m + \frac{t_m}{2} \cdot \text{tr} D_m \right) M_m\ , \\ M_m &= \sum_{g=1}^{\text{EltNr}} V_{m,g} \hat{M} {V_{m,g}}^{\top}\ . \end{aligned} \tag{69}$$

Thus together with the stiffness matrices K^{γ}, K^{χ} and K^{ζ} the vibration constraint can be written as

$$\inf_{\substack{(v,\eta)\in\mathcal{U}_h \\ \|(v,\eta)\|=1}} (v,\eta)^{\top} \left(\sum_{m=1}^{\text{EltNr}} K^{\gamma}(C_m) + K^{\chi}(C_m) + K^{\zeta}(D_m) - \hat{\lambda} M_m(C,D) \right) (v,\eta) \geq 0 \tag{70}$$

or in a more compact fashion as

$$\inf_{\substack{(v,\eta)\in\mathcal{U}_h \\ \|(v,\eta)\|=1}} (v,\eta)^{\top} \left(K_{\text{shell}}(C,D) - \hat{\lambda} M(C,D) \right) (v,\eta) \geq 0 \tag{71}$$

leading to the discrete Free Material Optimization problem for shells in the primal minimal weight formulation:

$$
\begin{array}{rcll}
\displaystyle\min_{\substack{(u,\theta)\in\mathcal{U}_h\\(C,D)\in\mathcal{C}_h}} \sum_{m=1}^{\mathrm{EltNr}} t_m\cdot\mathrm{tr}C_m+\frac{t_m}{2}\cdot\mathrm{tr}D_m & & & \\
\text{subject to} \qquad C_m\succeq 0\,;\ D_m & \succeq & 0 & \forall\, m=1,\dots,\mathrm{EltNr}\\
\begin{pmatrix} t_mC_m & 0\\ 0 & \frac{t_m}{2}D_m\end{pmatrix}-\varepsilon_m\mathbb{1}_5 & \succeq & 0 & \forall\, m=1,\dots,\mathrm{EltNr}\\
t_m\cdot\mathrm{tr}C_m+\dfrac{t_m}{2}\cdot\mathrm{tr}D_m & \leq & \rho^+ & \forall\, m=1,\dots,\mathrm{EltNr}\\
\left(\displaystyle\sum_{m=1}^{\mathrm{EltNr}} K^\gamma(C_m)+K^\chi(C_m)+K^\zeta(D_m)\right)(u,\theta) & = & f_h & \\
{f_h}^\top(u,\theta) & \leq & c & \\
\displaystyle\inf_{\substack{(v,\eta)\in\mathcal{U}_h\\\|(v,\eta)\|=1}} (v,\eta)^\top\left(K_{\mathrm{shell}}(C,D)-\hat{\lambda}M(C,D)\right)(v,\eta) & \geq & 0 &
\end{array}
$$

As in [36] this is a mathematical programming problem with linear matrix inequality constraints and standard nonlinear constraints, which can be turned into a semidefinite program and can be solved with the methods described in [36].

7. Free Material Optimization for Shells with Global Stability Constraints

The last type of constraints we investigate are global stability constraints. Constraints of this kind are used to avoid equilibrium states that can not be regarded as stable. Consider a shell in an equilibrium configuration (S_I) implying a displacement field $(u,\theta)\in\mathcal{U}$. As described in [4] there are three basic types of equilibrium states depending on the sign of the increase in total potential energy $P((u,\theta),(v,\eta))$, when the shell is subjected to sufficiently small displacements $(v,\eta)\in\mathcal{U}$ shifting it into the neighboring state (S_{II}). The increase in total potential energy is defined as

$$
P((u,\theta),(v,\eta)) = \Pi(u+v,\theta+\eta)-\Pi(u,\theta)\,. \tag{72}
$$

Using a taylor expansion for the first term this can also be written as

$$
\begin{aligned}
&P((u,\theta),(v,\eta)) =\\
&= \Pi(u,\theta)+\Pi'(u,\theta)(v,\eta)+\frac{1}{2}\Pi''(u,\theta)(v,\eta)^2+\|(v,\eta)\|^2\epsilon(v,\eta)-\Pi(u,\theta)\\
&= \underbrace{\Pi'(u,\theta)}_{=0\text{ in equilibrium}}(v,\eta)+\frac{1}{2}\Pi''(u,\theta)(v,\eta)^2+\|(v,\eta)\|^2\epsilon(v,\eta)\\
&= \frac{1}{2}\Pi''(u,\theta)(v,\eta)^2+\|(v,\eta)\|^2\epsilon(v,\eta)
\end{aligned}
$$

where $\lim_{(v,\eta)\to 0} \epsilon(v,\eta) = 0$. Thus the sign of the increase in total potential energy depends on the term $\Pi''(u,\theta)(v,\eta)^2$, which can be used to identify the equilibrium type as

- *stable equilibrium*, when the displacement field (u,θ) is a strict local minimum of the potential energy $\Pi(u,\theta)$. Under the assumption, that $\Pi(u,\theta)$ is twice continuously differentiable with respect to (u,θ), this is equivalent to the existence of a constant $c_1 > 0$ such that

$$\begin{aligned} \Pi'(u,\theta) &= 0 \quad \text{(equilibrium of } (S_I)) \\ \Pi''(u,\theta)(v,\eta)^2 &\geq c_1 \|(v,\eta)\|^2 \quad \forall\,(v,\eta) \in \mathcal{U} \end{aligned}$$

- *unstable equilibrium*, if there exists at least one displacement field $(v,\eta) \in \mathcal{U}$ such that

$$\begin{aligned} \Pi'(u,\theta) &= 0 \quad \text{(equilibrium of } (S_I)) \\ \Pi''(u,\theta)(v,\eta)^2 &< 0 \end{aligned}$$

- *neutral equilibrium*, which implies the following conditions

$$\begin{aligned} \Pi'(u,\theta) &= 0 \quad \text{(equilibrium of } (S_I)) \\ \exists\, c_2 > 0 \text{ such that } \quad \Pi''(u,\theta)(v,\eta)^2 &\geq 0 \quad \forall\,(v,\eta) \in \mathcal{U}\,,\ \|(v,\eta)\| \leq c_2 \\ \exists\,\{(v_i,\eta_i)\} \text{ such that } \Pi''(u,\theta)(v_i,\eta_i)^2 &= 0 \quad i = 1,2,\ldots \end{aligned} \tag{73}$$

  where $(v_i,\eta_i) \in \mathcal{U} \setminus \{0\}$. These specific displacements are called buckling modes of the structure.

It is our goal to investigate whether the structure will omit buckling behaviour. As a linear model does not contain the effect of buckling we have to expand our shell model using nonlinear terms to be able to describe these properties [31]. Buckling usually appears in a membrane state, hence nonlinear terms are only added to the membrane strains [27]. This results in the nonlinear membrane strains

$$\tilde{\gamma}_{\alpha\beta}(u) = \gamma_{\alpha\beta}(u) + \frac{1}{2} a^{\kappa\nu} \left(\gamma_{\kappa\alpha}(u) - \psi_{\kappa\alpha}(u)\right) \left(\gamma_{\nu\beta}(u) - \psi_{\nu\beta}(u)\right) + \frac{1}{2}\varphi_\alpha(u)\varphi_\beta(u) \tag{74}$$

where $\gamma_{\alpha\beta}(u)$ are the linear membrane strains already introduced in (15) and the remaining functions are defined as

$$\psi_{\kappa\alpha}(u) := \frac{1}{2}\left(u_{\alpha|\kappa} - u_{\kappa|\alpha}\right)\,, \tag{75}$$

$$\varphi_\alpha(u) := u_{3,\alpha} + b^\kappa_\alpha u_\kappa\,. \tag{76}$$

For the bending strains $\chi_{\alpha\beta}(u,\theta)$ and the shear strains $\zeta_\alpha(u,\theta)$ the linear formulas as given in (16) and (17) are used. Using these kinematic equations we follow the calculations presented in [4] and are hence able to calculate the increase of total potential energy $P((u,\theta),(v,\eta))$ as defined in (72)

$$\begin{aligned}
P((u,\theta),(v,\eta)) \;&=\; \frac{1}{2}\int_{\omega} tC^{\alpha\beta\lambda\mu}\tilde{\gamma}_{\alpha\beta}(u+s\cdot v)\tilde{\gamma}_{\lambda\mu}(u+s\cdot v) - tC^{\alpha\beta\lambda\mu}\tilde{\gamma}_{\alpha\beta}(u)\tilde{\gamma}_{\lambda\mu}(u)\\
&\quad + \frac{t^3}{12}C^{\alpha\beta\lambda\mu}\chi_{\alpha\beta}(u+s\cdot v,\theta+s\cdot\eta)\chi_{\lambda\mu}(u+s\cdot v,\theta+s\cdot\eta)\\
&\quad + tkD^{\alpha\lambda}\zeta_{\alpha}(u+s\cdot v,\theta+s\cdot\eta)\zeta_{\lambda}(u+s\cdot v,\theta+s\cdot\eta)\\
&\quad - \frac{t^3}{12}C^{\alpha\beta\lambda\mu}\chi_{\alpha\beta}(u,\theta)\chi_{\lambda\mu}(u,\theta) - tkD^{\alpha\lambda}\zeta_{\alpha}(u,\theta)\zeta_{\lambda}(u,\theta)\,d\mathcal{S}\\
&\quad - \int_{\omega} tf\cdot(u+s\cdot v) - tf\cdot u\,d\mathcal{S}\\
&\quad - \int_{\partial\omega_1} g_u\cdot(u+s\cdot v) + g_\theta\cdot(\theta+s\cdot\eta) - g_u\cdot u - g_\theta\cdot\theta\,dl
\end{aligned}$$

The potential energy has already been given in (24). Furthermore let $(u,\theta)\in\mathcal{U}$, $(v,\eta)\in\mathcal{U}$ and $s\in\mathbb{R}_+$. Employing the equilibrium condition $\Pi'(u,\theta)(v,\eta)\overset{!}{=}0$ leads to

$$\begin{aligned}
&\int_{\omega} tC^{\alpha\beta\lambda\mu}\Big(\gamma_{\alpha\beta}(u)\gamma_{\lambda\mu}(v) + \frac{1}{2}\,a^{\kappa\nu}a^{\sigma\tau}(\gamma_{\kappa\alpha}(u)-\psi_{\kappa\alpha}(u))(\gamma_{\nu\beta}(u)-\psi_{\nu\beta}(u))\,\cdot\\
&\qquad\cdot(\gamma_{\sigma\lambda}(u)-\psi_{\sigma\lambda}(u))(\gamma_{\tau\mu}(v)-\psi_{\tau\mu}(v)) + \frac{1}{2}\,\varphi_{\alpha}(u)\varphi_{\beta}(u)\varphi_{\lambda}(u)\varphi_{\mu}(v)\,+\\
&\quad+\gamma_{\alpha\beta}(u)\varphi_{\lambda}(u)\varphi_{\mu}(v) + \frac{1}{2}\,\gamma_{\alpha\beta}(v)\varphi_{\lambda}(u)\varphi_{\mu}(u) + \gamma_{\alpha\beta}(u)a^{\kappa\nu}(\gamma_{\kappa\lambda}(u)-\psi_{\kappa\lambda}(u))\,\cdot\\
&\qquad\cdot(\gamma_{\nu\mu}(v)-\psi_{\nu\mu}(v)) + \frac{1}{2}\,\gamma_{\alpha\beta}(u)a^{\kappa\nu}(\gamma_{\kappa\lambda}(u)-\psi_{\kappa\lambda}(u))(\gamma_{\nu\mu}(u)-\psi_{\nu\mu}(v))\,+\\
&\qquad\qquad+\varphi_{\alpha}(u)\varphi_{\beta}(u)a^{\kappa\nu}(\gamma_{\kappa\lambda}(u)-\psi_{\kappa\lambda}(u))(\gamma_{\nu\mu}(v)-\psi_{\nu\mu}(v))\,+\\
&\qquad\qquad+\varphi_{\alpha}(u)\varphi_{\beta}(v)a^{\kappa\nu}(\gamma_{\kappa\lambda}(u)-\psi_{\kappa\lambda}(u))(\gamma_{\nu\mu}(u)-\psi_{\nu\mu}(u))\Big)\,+\\
&\qquad\qquad+\frac{t^3}{12}\,C^{\alpha\beta\lambda\mu}\chi_{\alpha\beta}(u,\theta)\chi_{\lambda\mu}(v,\eta) + tkD^{\alpha\lambda}\zeta_{\alpha}(u,\theta)\zeta_{\lambda}(v,\eta)\,d\mathcal{S} \;=\\
&\qquad\qquad\;=\;\int_{\omega} tf\cdot v\,d\mathcal{S} + \int_{\partial\omega_1} g_u\cdot v - g_\theta\cdot\eta\,dl \quad \forall\,(v,\eta)\in\mathcal{U}
\end{aligned}$$

granting access to the quantity of interest

$$\begin{aligned}
\Pi''(u,\theta)(v,\eta)^2 \;&=\; \frac{1}{2}\int_{\omega} n^{\alpha\beta}(u)[a^{\kappa\nu}(\gamma_{\kappa\alpha}(v)-\psi_{\kappa\alpha}(v))(\gamma_{\nu\beta}(v)-\psi_{\nu\beta}(v)) + \varphi_{\alpha}(v)\varphi_{\beta}(v)]\\
&\quad + tC^{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}(v)\gamma_{\lambda\mu}(v) + \frac{t^3}{12}\,C^{\alpha\beta\lambda\mu}\chi_{\alpha\beta}(v,\eta)\chi_{\lambda\mu}(v,\eta)\\
&\quad + tkD^{\alpha\lambda}\zeta_{\alpha}(v,\eta)\zeta_{\lambda}(v,\eta)\,d\mathcal{S}\,, \qquad (77)
\end{aligned}$$

where we have used the definition $n^{\alpha\beta}(u) = tC^{\alpha\beta\lambda\mu}\tilde{\gamma}_{\alpha\beta}(u)$ for the initial stress term of the state (S_I). Throughout the buckling analysis it is assumed that the loads acting on the shell are given by $P+\lambda Q$, where P is a permanent load case applied to the initial configuration (S_I) and λQ is a fluctuating perturbation of the originally stable

equilibrium associated with P. The critical load $P + \lambda_c Q$ is defined as the load with the smallest value λ_c under which buckling occurs. The determination of the buckling modes is more accurate for a permanent load P, which is close to the critical load. In this case one may make a linear approximation of the stresses $n^{\alpha\beta} = p^{\alpha\beta} + \lambda q^{\alpha\beta}$, where $p^{\alpha\beta}(u_P)$ and $q^{\alpha\beta}(u_Q)$ are the stresses associated with the loads P and Q, respectively. This approximation, that allows for a linear calculation from the displacements u_P and u_Q and also the resulting stresses $p^{\alpha\beta}(u_P)$ and $q^{\alpha\beta}(u_Q)$ from the loads P and Q, is known as linear buckling.
Defining now the bilinear forms

$$\begin{aligned}
a((v,\eta),(w,\vartheta)) &:= \int_\omega tC^{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}(v)\gamma_{\lambda\mu}(w) + \frac{t^3}{12}C^{\alpha\beta\lambda\mu}\chi_{\alpha\beta}(v,\eta)\chi_{\lambda\mu}(w,\vartheta) \\
&\quad + tkD^{\alpha\lambda}\zeta_\alpha(v,\eta)\zeta_\lambda(w,\vartheta)\,d\mathcal{S} && (78)\\
b_{u_Q}((v,\eta),(w,\vartheta)) &:= \int_\omega q^{\alpha\beta}(u_Q)[a^{\kappa\nu}(\gamma_{\kappa\alpha}(v) - \psi_{\kappa\alpha}(v))(\gamma_{\nu\beta}(w) - \psi_{\nu\beta}(w)) \\
&\quad + \varphi_\alpha(v)\varphi_\beta(w)]\,d\mathcal{S} && (79)\\
c_{u_P}((v,\eta),(w,\vartheta)) &:= \int_\omega p^{\alpha\beta}(u_P)[a^{\kappa\nu}(\gamma_{\kappa\alpha}(v) - \psi_{\kappa\alpha}(v))(\gamma_{\nu\beta}(w) - \psi_{\nu\beta}(w)) \\
&\quad + \varphi_\alpha(v)\varphi_\beta(w)]\,d\mathcal{S} && (80)
\end{aligned}$$

it becomes apparent through comparison with the buckling condition (73) and the decisive term of the increase in total potential energy (77), that the buckling modes can be determined as eigenvectors (v_c, η_c) of the generalized eigenvalue problem

> *Find the eigenvalue of the smallest module λ_c and the associated eigenvector (v_c, η_c) which are solutions to the equation*

$$a((v_c,\eta_c),(w,\vartheta)) + c_{u_P}((v_c,\eta_c),(w,\vartheta)) + \lambda_c b_{u_Q}((v_c,\eta_c),(w,\vartheta)) = 0 \quad \forall\,(w,\vartheta) \in \mathcal{U}\,. \tag{81}$$

The corresponding load $P + \lambda_c Q$ is called the critical load of the shell.
We now want to add a constraint to the discretized Free Material Optimization problem in the minimum weight-formulation (37) to avoid buckling behavior. As in [17] we set $P = 0$ resulting in $u_P = 0$ and $p^{\alpha\beta} = 0$. We already know that the discretized form of $a((u,\theta),(v,\eta))$ is given by $(u,\theta)^\top K_{\text{shell}}(C,D)(v,\eta)$. Introducing the geometry stiffness matrix G_{shell} the term $b_{u_Q}((u,\theta),(v,\eta))$ can be rewritten in its discretized version as $(u,\theta)^\top G_{\text{shell}}(C,D,u,\theta)(v,\eta)$. Thus we are able to formulate a discretized global stability constraint by demanding that none of the eigenvalues of the generalized eigenvalue problem

$$K_{\text{shell}}(C,D)(v_c,\eta_c) + \lambda_c G_{\text{shell}}(C,D,u,\theta)(v_c,\eta_c) = 0 \tag{82}$$

lie in the interval $[0,1]$. According to [15] this condition can be reformulated as a matrix constraint of the form

$$K_{\text{shell}}(C,D) + G_{\text{shell}}(C,D,u,\theta) \succeq 0 \tag{83}$$

Hence we obtain the Free Material Optimization problem for shells with a global stability constraint

$$\begin{aligned} \min_{\substack{(u,\theta)\in\mathcal{U}_h\\(C,D)\in\mathcal{C}_h}} \sum_{m=1}^{\mathrm{EltNr}} t_m\cdot\mathrm{tr}C_m + \frac{t_m}{2}\cdot\mathrm{tr}D_m & & & \qquad (84)\\ \text{subject to} \qquad C_m\succeq 0\,;\; D_m &\succeq& 0 & \quad\forall\, m=1,\dots,\mathrm{EltNr}\\ \begin{pmatrix} t_mC_m & 0\\ 0 & \frac{t_m}{2}D_m\end{pmatrix} - \varepsilon_m\mathbb{1}_5 &\succeq& 0 & \quad\forall\, m=1,\dots,\mathrm{EltNr}\\ t_m\cdot\mathrm{tr}C_m + \frac{t_m}{2}\cdot\mathrm{tr}D_m &\le& \rho^+ & \quad\forall\, m=1,\dots,\mathrm{EltNr}\\ \left(\sum_{m=1}^{\mathrm{EltNr}} K^\gamma(C_m)+K^\chi(C_m)+K^\zeta(D_m)\right)(u,\theta) &=& f_h & \\ {f_h}^\top(u,\theta) &\le& c & \\ K_{\mathrm{shell}}(C,D)+G_{\mathrm{shell}}(C,D,u,\theta) &\succeq& 0 & \end{aligned}$$

which is a nonlinear SDP with nonlinear matrix constraints.

A. List of Frequently Used Symbols

Symbol	Name or description	Place of definition or first occurence
a	determinant of first fundamental form $a_{\alpha\beta}$	(3)
a_i	covariant basis vectors of the surface	(1)
$a_{\alpha\beta}$	first fundamental form of a surface	(2)
A_m^γ	discretized dyadic strain matrix for the strain γ in the element m	(42)
A_m^χ	discretized dyadic strain matrix for the strain χ in the element m	(43)
A_m^ζ	discretized dyadic strain matrix for the strain ζ in the element m	(44)
$b_{\alpha\beta}$	second fundamental form of a surface	(7)
B_i^γ	discretized strain matrix for the strain γ at node i	(39)
B_i^χ	discretized strain matrix for the strain χ at node i	(40)
B_i^ζ	discretized strain matrix for the strain ζ at node i	(41)
$c_{\alpha\beta}$	third fundamental form of a surface	(8)
$C^{\alpha\beta\lambda\mu}$	elasticity tensor for membrane and bending	(18)
C	matrix notation of the elasticity tensor $C^{\alpha\beta\lambda\mu}$	(22)
C_{DC}	coefficient matrix for displacement constraints	(51)
$\tilde{\mathcal{C}}$	set of sym. positive semidefinite matrices	(30)

Symbol	Name or description	Place of definition or first occurence
d_{DC}	right hand side for displacement constraints	(51)
$D^{\alpha\lambda}$	elasticity tensor for shear	(18)
D	matrix notation of the elasticity tensor $D^{\alpha\lambda}$	(22)
f	external force resultant density	(24)
g_u	external traction resultant density	(24)
g_θ	external moment resultant density	(24)
$\mathfrak{H}$	mean curvature of the midsurface	(9)
$\mathfrak{K}$	Gaussian curvature of the midsurface	(10)
k	shear correction factor	(18)
m^λ	transverse shear force resultant (tensor notation)	(18)
m	transverse shear force resultant (vector notation)	(21)
$M^{\lambda\mu}$	moment resultant (tensor notation)	(18)
M	moment resultant (vector notation)	(21)
$N^{\lambda\mu}$	force resultant (tensor notation)	(18)
N	force resultant (vector notation)	(21)
P	increase in total potential energy	(72)
$\mathcal{R}$	shell body in three dimensions	Sect. 2.2
s_e^{ip}	upper bound for in-plane strains	(54)
s_e^{oop}	upper bound for out-of-plane strains	(54)
s_σ^{ip}	upper bound for in-plane stresses	(53)
s_σ^{oop}	upper bound for out-of-plane stresses	(53)
$\mathcal{S}$	midsurface of the shell	Sect. 2.1
t	thickness of the shell	Sect. 2.2
u	translational displacement	Sect. 2.2
$\mathcal{U}$	set of admissible displacements	(14)
V	upper global bound on the trace of C and D	(34)
α	Lagrange multiplier for volume constraint	Sect. 2.4 (P_P)
β_l	Lagrange multiplier for lower box constraint	Sect. 2.4 (P_P)
β_u	Lagrange multiplier for upper box constraint	Sect. 2.4 (P_P)
$\gamma_{\alpha\beta}$	membrane strain (tensor notation)	(15)
γ	membrane strain (vector notation)	(20)
$\Gamma^\lambda_{\alpha\mu}$	Christoffel symbol of a surface	(5)
$\zeta_{\alpha\beta}$	shear strain (tensor notation)	(17)
ζ	shear strain (vector notation)	(20)
θ	rotational displacement	Sect. 2.2
$\vartheta_i(r,s)$	bilinear 2D Lagrange shape function	Sect. 3.1
λ	eigenvalue of the vibration problem	(63)
ξ	space variable $\xi = (\xi_1, \xi_2, \xi_3)^\top$	(13)
$\Pi(u,\theta)$	potential energy of a shell	(24)
ρ^+	upper local bound on the trace of C and D	(35)
ρ^-	lower local bound on the trace of C and D	(35)

Symbol	Name or description	Place of definition or first occurence
τ	time variable	(55)
$\chi_{\alpha\beta}$	bending strain (tensor notation)	(16)
χ	bending strain (vector notation)	(20)
ω	reference domain for the midsurface $\mathcal{S}$	Sect. 2.1
ω_m	quadrangular element of the midsurface mesh	Sect. 3.1
$\partial\omega$	Lipschitz boundary of midsurface ω	Sect. 2.2
$\partial\omega_0$	clamped part of the boundary $\partial\omega$	Sect. 2.2
$\partial\omega_1$	free part of the boundary $\partial\omega$	Sect. 2.2
$(\,)_{,\mu}$	partial differentiation with respect to surface coordinates	Sect. 2.1
$(\,)_{\vert\mu}$	covariant differentiation with respect to first fundamental form of a surface	(6)

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