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FREE MATERIAL OPTIMIZATION FOR PLATES AND SHELLS: THE SINGLE LOAD CASE *

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Abstract

In this article, we present the Free Material Optimization (FMO) problem for plates and shells based on Naghdi's shell model. In FMO – a branch of structural optimization – we search for the ultimately best material properties in a given design domain loaded by a set of given forces. The optimization variable is the full material tensor at each point of the design domain. We give a basic formulation of the problem and prove existence of an optimal solution. Lagrange duality theory allows to identify the basic problem as the dual of an infinite-dimensional convex nonlinear semidefinite program. After discretization by the finite element method the latter problem can be solved using a nonlinear SDP code. The article is concluded by several numerical studies.

Keywords: Free Material Optimization, Structural Optimization, Shells, Continuum Mechanics, Elasticity.

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1 Introduction

Structural optimization deals with the problem of finding the stiffest structure subjected to a set of given loads and boundary conditions, when only a limited amount of material resources is available. Nowadays, this approach plays an important role in the construction of light-weight structures like airplanes and cars. Large parts of these objects as, for instance, the fuselage, consist of thin-walled structures like shells and plates. This is the reason why structural optimization of shells has received a lot of attention in the design optimization community over the last couple of years. For example, shape optimization techniques have been used to vary the geometry and boundary of a shell with the goal to stiffen the structure [6]. Various approaches try to identify the optimal topology of a shell in the sense of 0–1–designs. For an overview in the case of plates see [4]. On spherical shells it is possible to calculate the topological derivative and to exploit this information with the purpose of finding the optimal position of holes [18]. Only recently, free sizing optimization taking strength and stability constraints into account has been used to improve the design of shell structures [8].

Another important class of shell design problems is based on material optimization. Here the design variables reflect not only the distribution of material in the design domain, but also the local properties of the material. The methods used in the area of material optimization differ in the choice of the admissible set of materials. In [19] a pseudo density of the material is varied using a SIMP-approach. Rather than cutting the solution space down to 0–1–designs the author proposes to realize the optimal solution using foams that can be produced in manifold densities. In aerospace industry the use of composite materials is very common. In [25] the authors suggest to design composite shells by optimization of the material selection and fibre angles in a laminated shell structure. It is even possible to consider fully anisotropic elasticity tensors as admissible set for the design optimization as shown for Reissner-Mindlin plates in [2]. Finally, there are approaches taking advantage of adaptive methods – either by changing the parametrization of the design space during optimization or by adapting the model via switching between shape and material optimization; see [21].

In this article, we focus on Free Material Optimization, originally introduced for the optimal design of solid bodies by [3]. The design variable used in Free Material Optimization is the full material tensor at each point of the design domain. Therefore it yields not only the optimal material distribution, but also the material properties at each point. Various solution techniques for this problem have been proposed; see, for example, [27]. Due to the high freedom in the design space the resulting material/structure is typically hard to manufacture. Nevertheless it gives valuable information about the optimal material density, symmetry and principal directions, which can be exploited to realize approximations of the optimal design. One possible realization by tapelayering is described in [14]. In the recent years, the formulation of the Free Material Optimization problem has been extended to cover multiple load cases [1], stability control by consideration of global buckling [16] and stress constraints [17]. In this article we propose a formulation of Free Material Optimization based on the linear elastic shell model of Naghdi [22] suited for thin-walled structures like airplanes, cars and pipes.

Table of frequently used symbols

Symbol	Name or description	Place of definition or first occurence
a_i	covariant basis vectors of the surface	(1)
$a_{\alpha\beta}$	first fundamental form of a surface	Sect. 2
B_i^{γ}	discretized strain matrix for the strain γ at node i	(64)
$b_{\alpha\beta}$	second fundamental form of a surface	Sect. 2
A_m^{γ}	discretized dyadic strain matrix for the strain γ in the element m	(65)
$c_{\alpha\beta}$	third fundamental form of a surface	Sect. 2
$\mathcal{C}$	set of admissible elasticity matrices	(3)
$\tilde{\mathcal{C}}$	set of sym. positive semidefinite matrices	(38)
$C^{\alpha\beta\lambda\mu}$	elasticity tensor for membrane and bending	Sect. 2
C	matrix notation of the elasticity tensor $C^{\alpha\beta\lambda\mu}$	(10)
$D^{\alpha\lambda}$	elasticity tensor for shear	Sect. 2
D	matrix notation of the elasticity tensor $D^{\alpha\lambda}$	(10)
f	external force resultant density	Sect. 2
g	external traction resultant density	Sect. 2
h	external moment resultant density	Sect. 2
$\mathcal{L}$	set of admissible Lagrange multipliers	(39)
$\mathcal{M}$	set of sym. positive semidefinite matrices with bounded trace	(72)
m^{λ}	transverse shear force resultant (tensor notation)	(7)
m	transverse shear force resultant (vector notation)	(9)
$M^{\lambda\mu}$	moment resultant (tensor notation)	(7)
M	moment resultant (vector notation)	(9)
$N^{\lambda\mu}$	force resultant (tensor notation)	(7)
N	force resultant (vector notation)	(9)
t	thickness of the shell	Sect. 2
u	translational displacement	Sect. 2
$\mathcal{U}$	set of admissible displacements	(5)
V	upper global bound on the trace of C and D	(19)
$v_{\alpha,\mu}$	partial derivative of v_α w.r.t. ξ^μ	Sect. 2
$v_{\alpha\|\mu}$	surface covariant derivative of v_α	(2)
α	Lagrange multiplier for volume constraint	Sect. 4
β_l	Lagrange multiplier for lower box constraint	Sect. 4
β_u	Lagrange multiplier for upper box constraint	Sect. 4

Symbol	Name or description	Place of definition or first occurence
$\gamma_{\alpha\beta}$	membrane strain (tensor notation)	(6)
γ	membrane strain (vector notation)	(8)
$\Gamma^{\lambda}_{\alpha\mu}$	Christoffel symbol of a surface	(3)
$\delta_x J(x,y)(\delta x)$	Gâteaux derivative of $J(x,y)$ w.r.t. x	Sect. 4
$\zeta_{\alpha\beta}$	shear strain (tensor notation)	(6)
ζ	shear strain (vector notation)	(8)
θ	rotational displacement	Sect. 2
$\Pi(u,\theta)$	potential energy of a shell	(12)
ρ^{+}	upper local bound on the trace of C and D	(20)
ρ^{-}	lower local bound on the trace of C and D	(20)
$\chi_{\alpha\beta}$	bending strain (tensor notation)	(6)
χ	bending strain (vector notation)	(8)
χ_{ω}	characteristic function of the set ω	(58)
ω	midsurface of the shell	Sect. 2
$\partial\omega$	Lipschitz boundary of midsurface ω	Sect. 2
$\partial\omega_0$	clamped part of the boundary $\partial\omega$	Sect. 2
$\partial\omega_1$	free part of the boundary $\partial\omega$	Sect. 2
$\langle\cdot,\cdot\rangle$	inner product for matrices	Sect. 4

2 Naghdi's Shell Model

We start with a mathematical description of Naghdi's shell model [22, 9, 10]. The geometry of a Naghdi shell is described by the midsurface ω – an open bounded two-dimensional set in Euclidean space, which can be parametrized by a sufficiently smooth function $\Phi : \mathbb{R}^2 \to \mathbb{R}^3$, $\Phi \in W^{2,\infty}(\omega)$.

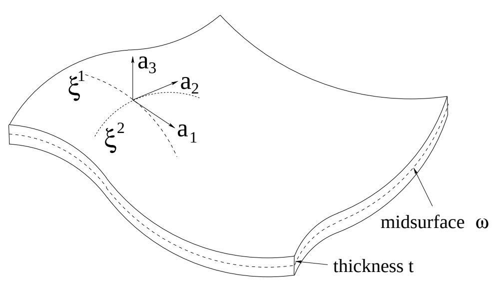

Fig. 1: Curvilinear coordinates on the midsurface

Hence it is advantageous to use curvilinear coordinates denoted by ξ^i (in accordance with common notation in shell theory Latin indices run over 1, 2 and 3, while Greek indices run only over 1 and 2). The covariant basis vectors are then defined by

$$a_\alpha = \frac{\partial \Phi}{\partial \xi^\alpha} \quad , \quad a_3 = \frac{a_1 \times a_2}{\|a_1 \times a_2\|} \,. \tag{1}$$

Moreover, the surface covariant derivative of a vector field v is given by

$$v_{\alpha|\mu} = v_{\alpha,\mu} - \Gamma^{\lambda}_{\alpha\mu} v_\lambda \,, \tag{2}$$

where $v_{\alpha,\mu}$ is the partial derivative of v_α with respect to ξ^μ and $\Gamma^{\lambda}_{\alpha\mu}$ is the Christoffel symbol of the midsurface

$$\Gamma^{\lambda}_{\alpha\mu} = a_{\alpha,\mu} a^{\lambda} \,. \tag{3}$$

Furthermore the fundamental forms of the midsurface are defined by

- first fundamental form: $a_{\alpha\beta} = a_\alpha a_\beta$,
- second fundamental form: $b_{\alpha\beta} = -a_{3,\beta} a_\alpha$,
- third fundamental form: $c_{\alpha\beta} = b_\alpha^\lambda b_{\lambda\beta}$.

It turns out that the midsurface alone contains not enough information to describe bending and shear effects. A remedy is provided by the theory of Cosserat continua: at each point $x \in \omega$ a director vector d is attached to the shell, adding the lacking degrees of freedom [11, 24]. These director vectors can be interpreted as material lines along the thickness of the shell. The deformation of the loaded shell can be described by a translation of all points of the midsurface $u \in \left[H^1(\omega)\right]^3$ and a rotation of the associated director vectors $\theta \in \left[H^1(\omega)\right]^2$. Neglecting rotations of the director vectors around their own axis we obtain the following displacement formula:

$$U(\xi^1, \xi^2, \xi^3) = u(\xi^1, \xi^2) + \xi^3 \theta_\lambda(\xi^1, \xi^2) a^\lambda(\xi^1, \xi^2) \,. \tag{4}$$

In the remainder of this article, we consider a shell with a Lipschitz boundary $\partial\omega$. The shell is clamped at parts of the boundary. To this end we partition $\partial\omega$ into two sets $\partial\omega_0$ and $\partial\omega_1$ which are open in $\partial\omega$, $\partial\omega = \overline{\partial\omega_0} \cup \overline{\partial\omega_1}$ and $\partial\omega_0 \cap \partial\omega_1 = \emptyset$. Then Dirichlet boundary conditions are applied on $\partial\omega_0$ and the shell is subjected to forces and moments on $\partial\omega_1$. Using this, we define the set of admissible displacements to be

$$\mathcal{U} := \left\{ (u, \theta) \in \left[H^1(\omega)\right]^5 \middle| \, \theta \cdot a_3 = 0 \,;\; u = \theta = 0 \text{ on } \partial\omega_0 \right\} . \tag{5}$$

Note that all statements given in this paper are also true if only some components of (u, θ) are fixed at the boundary $\partial\omega_0$. As a consequence we obtain $\left[H_0^1(\omega)\right]^5 \subset \mathcal{U} \subset \left[H^1(\omega)\right]^5$. It is now possible to deduce formulas for membrane strains $\gamma_{\alpha\beta}$, bending strains $\chi_{\alpha\beta}$ and shear strains ζ_α, respectively:

$$\begin{aligned}
\gamma_{\alpha\beta}(u) &= \frac{1}{2}\left(u_{\alpha|\beta} + u_{\beta|\alpha}\right) - b_{\alpha\beta} u_3 \,, \\
\chi_{\alpha\beta}(u, \theta) &= \frac{1}{2}\left(\theta_{\alpha|\beta} + \theta_{\beta|\alpha} - b_\beta^\lambda u_{\lambda|\alpha} - b_\alpha^\lambda u_{\lambda|\beta}\right) + c_{\alpha\beta} u_3 \,, \\
\zeta_\alpha(u, \theta) &= \frac{1}{2}\left(\theta_\alpha + u_{3,\alpha} + b_\alpha^\lambda u_\lambda\right) \,.
\end{aligned} \tag{6}$$

The assumption of linear elasticity in Naghdi's shell model leads to the following Hooke's law:

$$\begin{aligned}
N^{\lambda\mu} &= t\, C^{\lambda\mu\alpha\beta} \gamma_{\alpha\beta} \,, \\
M^{\lambda\mu} &= \frac{t^3}{12} C^{\lambda\mu\alpha\beta} \chi_{\alpha\beta} \,, \\
m^\lambda &= t\, D^{\lambda\alpha} \zeta_\alpha \,.
\end{aligned} \tag{7}$$

Here $C^{\lambda\mu\alpha\beta}$ and $D^{\lambda\alpha}$ are the elasticity tensors of the shell. $C^{\lambda\mu\alpha\beta}$ is a fourth-order tensor with the following symmetries:

$$C^{\lambda\mu\alpha\beta} = C^{\mu\lambda\alpha\beta} \quad , \quad C^{\lambda\mu\alpha\beta} = C^{\lambda\mu\beta\alpha} \quad , \quad C^{\lambda\mu\alpha\beta} = C^{\alpha\beta\lambda\mu} \, .$$

$D^{\lambda\alpha}$ is a symmetric second order tensor satisfying $D^{\lambda\alpha} = D^{\alpha\lambda}$. Moreover, the symmetric second order tensors $N^{\lambda\mu}$ and $M^{\lambda\mu}$ are called force resultant and moment resultant, respectively, and m^{λ} is the transverse shear force resultant. Finally t is the thickness of the shell. In the following we assume the thickness of the shell to be constant. Note however that the main results presented in this article remain valid for a thickness profile $t = t(x)$, which remains unchanged during optimization. The symmetry of the tensors allows us to rewrite Hooke's law using the following vectors and matrices:

$$\gamma = \begin{pmatrix} \gamma_{11} \\ \gamma_{22} \\ \sqrt{2}\gamma_{12} \end{pmatrix}, \qquad \chi = \begin{pmatrix} \chi_{11} \\ \chi_{22} \\ \sqrt{2}\chi_{12} \end{pmatrix}, \qquad \zeta = \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix}, \tag{8}$$

$$N = \begin{pmatrix} N_{11} \\ N_{22} \\ \sqrt{2}N_{12} \end{pmatrix}, \qquad M = \begin{pmatrix} M_{11} \\ M_{22} \\ \sqrt{2}M_{12} \end{pmatrix}, \qquad m = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}, \tag{9}$$

$$C = \begin{pmatrix} C_{1111} & C_{1122} & \sqrt{2}C_{1112} \\ C_{1122} & C_{2222} & \sqrt{2}C_{2212} \\ \sqrt{2}C_{1112} & \sqrt{2}C_{2212} & 2C_{1212} \end{pmatrix}, \qquad D = \begin{pmatrix} D_{11} & D_{12} \\ D_{12} & D_{22} \end{pmatrix}. \tag{10}$$

Then Hooke's law takes the form

$$\begin{aligned} N(x) &= t\, C(x)\gamma(u(x))\,, \\ M(x) &= \frac{t^3}{12}\, C(x)\chi(u(x),\theta(x))\,, \\ m(x) &= t\, D(x)\zeta(u(x),\theta(x)) \end{aligned} \tag{11}$$

and the potential energy $\Pi(u,\theta)$ of the Naghdi shell can be written as

$$\Pi(u,\theta) = \frac{1}{2}\int_{\omega} t\gamma^{\top}C\gamma + \frac{t^3}{12}\chi^{\top}C\chi + t\zeta^{\top}D\zeta\, dS - \int_{\omega} tf\cdot u\, dS - \int_{\partial\omega_1} g\cdot u + h\cdot\theta\, dl\,, \tag{12}$$

where $f \in \left[L^2(\omega)\right]^3$ is a given force resultant density and $g \in \left[L^2(\partial\omega_1)\right]^3$ and $h \in \left[L^2(\partial\omega_1)\right]^2$ are given traction and moment resultant densities, respectively. The shell is in equilibrium for any $(u,\theta) \in \mathcal{U}$ that minimizes the potential energy

$$\min_{(u,\theta)\in\mathcal{U}} \Pi(u,\theta)\,. \tag{13}$$

It is also possible to treat plates in this context. Assuming a planar midsurface allows to deduce the Reissner-Mindlin plate model from Naghdi's shell model. A planar midsurface has no curvature and thus a constant normal vector a_3. This results in vanishing second and third fundamental forms of the midsurface ω:

$$b_{\alpha\beta} = 0 \quad , \quad c_{\alpha\beta} = 0 \,. \tag{14}$$

In this case the formulas for the strains boil down to:

$$\begin{aligned}
\gamma_{\alpha\beta}(u_1, u_2) &= \frac{1}{2}\left(u_{\alpha|\beta} + u_{\beta|\alpha}\right) , \\
\chi_{\alpha\beta}(\theta) &= \frac{1}{2}\left(\theta_{\alpha|\beta} + \theta_{\beta|\alpha}\right) , \\
\zeta_\alpha(u_3, \theta) &= \frac{1}{2}\left(\theta_\alpha + u_{3,\alpha}\right) .
\end{aligned} \tag{15}$$

The equilibrium state of the plate is again found by minimizing the potential energy

$$\begin{aligned}
\min_{(u,\theta)\in\mathcal{U}} \Pi(u, \theta) = \; & \frac{1}{2}\int_\omega t\gamma^\top(u_1, u_2)C\gamma(u_1, u_2) + \frac{t^3}{12}\chi^\top(\theta)C\chi(\theta) \\
& + t\zeta^\top(u_3, \theta)D\zeta(u_3, \theta)\, dS - \int_\omega tf \cdot u\, dS - \int_{\partial\omega_1} g \cdot u + h \cdot \theta\, dl \,.
\end{aligned} \tag{16}$$

When solving the elasticity problem for a plate this can be separated into the membrane problem

$$\min_{(u_1,u_2)\in\mathcal{U}} \frac{1}{2}\int_\omega t\gamma^\top(u_1, u_2)C\gamma(u_1, u_2)\, dS - \int_\omega t\left(f_1 u_1 + f_2 u_2\right) dS - \int_{\partial\omega_1} \left(g_1 u_1 + g_2 u_2\right) dl$$

and the so-called Reissner-Mindlin problem

$$\min_{(u_3,\theta)\in\mathcal{U}} \frac{1}{2}\int_\omega \frac{t^3}{12}\chi^\top(\theta)C\chi(\theta) + t\zeta^\top(u_3, \theta)D\zeta(u_3, \theta)\, dS - \int_\omega tf_3 u_3\, dS - \int_{\partial\omega_1} g_3 u_3 + h \cdot \theta\, dl \,.$$

3 The Single Load Problem

Up to now we have merely described the physical behavior of the shell. However our overall goal is to find the stiffest structure which is subjected to a given set of loads f, g and h. A measure on how much a structure will deform under these loads is given by the compliance

$$\begin{aligned}
\mathrm{comp}(C, D) &= -\min_{(u,\theta)\in\mathcal{U}} \Pi_{C,D}(u, \theta) = \max_{(u,\theta)\in\mathcal{U}} -\Pi_{C,D}(u, \theta) \\
&= \max_{(u,\theta)\in\mathcal{U}} -\frac{1}{2}\int_\omega t\gamma^\top C\gamma + \frac{t^3}{12}\chi^\top C\chi + t\zeta^\top D\zeta\, dS \\
&\qquad + \int_\omega tf \cdot u\, dS + \int_{\partial\omega_1} g \cdot u + h \cdot \theta\, dl \,.
\end{aligned} \tag{17}$$

Apparently the compliance is given by the negative potential energy in equilibrium. In order to find the stiffest structure possible we now minimize the compliance with respect to the design variables. As we intend to work with Free Material Optimization these variables are the full elasticity tensors C and D. We want to allow for holes and material-no-material situations in the optimal structures, therefore we choose $C \in [L^\infty(\omega)]^{3\times 3}$ and $D \in [L^\infty(\omega)]^{2\times 2}$. As pointed out in section 2 the matrices have to be symmetric, furthermore they also have to be positive semidefinite as they describe a physical material:

$$C = C^\top \succeq 0 \quad , \quad D = D^\top \succeq 0\,. \tag{18}$$

As a measure for the amount of material used at a certain point $x \in \omega$ we simply use the summed traces of the matrices $\mathrm{tr}(C) + \frac{1}{2}\mathrm{tr}(D)$. The factor $\frac{1}{2}$ is necessary to be able to compare the results with the three–dimensional solid case. As we want to limit the material resources, we add the volume constraint

$$\int_\omega t\left(\mathrm{tr}(C) + \frac{1}{2}\mathrm{tr}(D)\right) dS \le V\,. \tag{19}$$

Finally we add box constraints to avoid arbitrarily high material concentrations at single points:

$$0 \le \rho^- \le t\,\mathrm{tr}(C(x)) + \frac{1}{2}\,t\,\mathrm{tr}(D(x)) \le \rho^+\,. \tag{20}$$

Summarizing (18), (19) and (20) we obtain the set of admissible elasticity tensors

$$\mathcal{C} := \left\{ (C,D) \in [L^\infty(\omega)]^{3\times 3} \times [L^\infty(\omega)]^{2\times 2} \;\middle|\; \begin{array}{l} C = C^\top \succeq 0 \\ D = D^\top \succeq 0 \\ \int_\omega t\left(\mathrm{tr}(C(x)) + \frac{1}{2}\mathrm{tr}(D(x))\right)\, dS \le V \\ 0 \le \rho^- \le t\,\mathrm{tr}(C(x)) + \frac{1}{2}\,t\,\mathrm{tr}(D(x)) \le \rho^+ \end{array} \right\}$$

For simplicity of notation we will assume $\rho^- = 0$. But note that all statements presented in this paper are also true for positive ρ^-. We finally are able to formulate the single load problem for shells:

$$\begin{aligned} \min_{(C,D)\in\mathcal{C}} \mathrm{comp}(C,D) \quad = \quad & \min_{(C,D)\in\mathcal{C}} \max_{(u,\theta)\in\mathcal{U}} -\frac{1}{2}\int_\omega t\gamma^\top C\gamma + \frac{t^3}{12}\chi^\top C\chi + t\zeta^\top D\zeta\, dS \\ & + \int_\omega tf\cdot u\, dS + \int_{\partial\omega_1} g\cdot u + h\cdot\theta\, dl\,. \end{aligned} \tag{21}$$

Introducing the function

$$J((C,D),(u,\theta)) := -\frac{1}{2}\int_\omega t\gamma^\top C\gamma + \frac{t^3}{12}\chi^\top C\chi + t\zeta^\top D\zeta\, dS + \int_\omega tf\cdot u\, dS + \int_{\partial\omega_1} g\cdot u + h\cdot\theta\, dl$$

we rewrite the latter optimization problem as

$$\min_{(C,D)\in\mathcal{C}} \max_{(u,\theta)\in\mathcal{U}} J((C,D),(u,\theta))\,. \tag{22}$$

In the case of plates we start from the equilibrium problem (16). The uncoupling into the membrane and the Reissner–Mindlin problem is not possible anymore when working with Free Material Optimization, as the material tensor C is one of the optimization variables connecting the membrane and bending terms. Thus the single load problem for Reissner–Mindlin plates takes the form

$$\begin{aligned}\min_{(C,D)\in\mathcal{C}} \max_{(u,\theta)\in\mathcal{U}} J((C,D),(u,\theta)) \quad := \quad & -\frac{1}{2}\int_\omega t\gamma^\top C\gamma + \frac{t^3}{12}\chi^\top C\chi + t\zeta^\top D\zeta \, dS \\ & + \int_\omega tf\cdot u\, dS + \int_{\partial\omega_1} g\cdot u + h\cdot\theta\, dl\,. \end{aligned} \tag{23}$$

This problem has already been formulated by Bendsœ and Díaz, who propose a solution via analytic derivation of the optimal material properties [2].

We now want to show existence of optimal solutions for problem (22). It can be easily seen that an optimal solution of the single load problem for shells is a saddle-point of the functional $J((C,D),(u,\theta))$. Thus existence of an optimal point follows from a Minimax-Theorem.

Theorem 1. *Problem (22) has an optimal solution* $((C^*,D^*),(u^*,\theta^*)) \in \mathcal{C}\times\mathcal{U}$.

Proof. $\mathcal{C} \subseteq [L^\infty(\omega)]^{3\times3} \times [L^\infty(\omega)]^{2\times2}$ is a convex, non-empty set for appropriately chosen constants V and ρ^+. As it lies in a norm ball of $[L^\infty(\omega)]^{3\times3} \times [L^\infty(\omega)]^{2\times2}$, it is weak*–compact and thus closed and bounded in the weak*–topology. $\mathcal{U}$ on the other hand is convex, closed and non-empty in the $\left[H^1(\omega)\right]^5$–topology. It is then according to the Minimax-Theorem shown in [20] sufficient to show

(i) for all $(C,D)\in\mathcal{C}$: $(u,\theta)\mapsto J((C,D),(u,\theta))$ is concave and continuous,

(ii) for all $(u,\theta)\in\mathcal{U}$: $(C,D)\mapsto J((C,D),(u,\theta))$ is convex and continuous,

(iii) there exist $(C_0,D_0)\in\mathcal{C}$ such that

$$J((C_0,D_0),(u,\theta)) \longrightarrow -\infty \qquad \forall\,(u,\theta)\in\mathcal{U} \text{ with } \|(u,\theta)\|_{H^1(\omega)} \longrightarrow \infty$$

in order to prove the existence of at least one saddle-point $((C^*,D^*),(u^*,\theta^*)) \in \mathcal{C}\times\mathcal{U}$ of J. Note that the original form of the theorem in [20] can be obtained by multiplying $J((C,D),(u,\theta))$ with -1.

Using the Schwarz inequality we get

$$\begin{aligned}
\big|J((C,D),(u,\theta))\big| &= \Big| -\frac{1}{2}\int_\omega t\gamma^\top C\gamma + \frac{t^3}{12}\chi^\top C\chi + t\zeta^\top D\zeta \, dS \\
&\qquad + \int_\omega tf\cdot u\, dS + \int_{\partial\omega_1} g\cdot u + h\cdot\theta\, dl \Big| \\
&\le \left|\frac{1}{2}\int_\omega t\gamma^\top C\gamma\, dS\right| + \left|\frac{1}{2}\int_\omega \frac{t^3}{12}\chi^\top C\chi\, dS\right| + \left|\frac{1}{2}\int_\omega t\zeta^\top D\zeta\, dS\right| \\
&\qquad + \left|\int_\omega tf\cdot u\, dS\right| + \left|\int_{\partial\omega_1} g\cdot u\, dl\right| + \left|\int_{\partial\omega_1} h\cdot\theta\, dl\right| \\
&\le \frac{t}{2}\|C\|_{L^\infty(\omega)}\|\gamma\|^2_{L^2(\omega)} + \frac{t^3}{24}\|C\|_{L^\infty(\omega)}\|\chi\|^2_{L^2(\omega)} + \frac{t}{2}\|D\|_{L^\infty(\omega)}\|\zeta\|^2_{L^2(\omega)} \\
&\quad + t\|f\|_{L^2(\omega)}\|u\|_{H^1(\omega)} + \|g\|_{L^2(\partial\omega_1)}\|u\|_{H^1(\partial\omega_1)} + \|h\|_{L^2(\partial\omega_1)}\|\theta\|_{H^1(\partial\omega_1)}\,.
\end{aligned}$$

As $\Phi \in W^{2,\infty}(\omega)$ we have $a_{\alpha\beta} \in W^{1,\infty}(\omega)$ and thus $\Gamma^{\alpha\mu}_\lambda \in L^\infty(\omega)$ and $b_{\alpha\beta} \in L^\infty(\omega)$. This implies

$$\begin{aligned}
\|\gamma\|^2_{L^2(\omega)} &= \|\gamma_{11}\|^2_{L^2(\omega)} + \|\gamma_{22}\|^2_{L^2(\omega)} + 2\|\gamma_{12}\|^2_{L^2(\omega)} \\
&= \|u_{1|1} - b_{11}u_3\|^2_{L^2(\omega)} + \|u_{2|2} - b_{22}u_3\|^2_{L^2(\omega)} + 2\left\|\frac{1}{2}u_{1|2} + \frac{1}{2}u_{2|1} - b_{12}u_3\right\|^2_{L^2(\omega)} \\
&= \|u_{1,1} - \Gamma^1_{11}u_1 - \Gamma^2_{11}u_2 - b_{11}u_3\|^2_{L^2(\omega)} + \|u_{2,2} - \Gamma^1_{22}u_1 - \Gamma^2_{22}u_2 - b_{22}u_3\|^2_{L^2(\omega)} \\
&\quad + \frac{1}{2}\left\|u_{1,2} - \Gamma^1_{12}u_1 - \Gamma^2_{12}u_2 + u_{2,1} - \Gamma^1_{21}u_1 - \Gamma^2_{21}u_2 - 2b_{12}u_3\right\|^2_{L^2(\omega)} \\
&\le \|u_{1,1}\|_{L^2(\omega)} + \|\Gamma^1_{11}\|_{L^\infty(\omega)}\|u_1\|_{L^2(\omega)} + \|\Gamma^2_{11}\|_{L^\infty(\omega)}\|u_2\|_{L^2(\omega)} + \|b_{11}\|_{L^\infty(\omega)}\|u_3\|_{L^2(\omega)}\dots \\
&\le c_\gamma\|u\|_{H^1(\omega)}\,.
\end{aligned}$$

with $c_\gamma > 0$. Analogously we conclude $\|\chi\|^2_{L^2(\omega)} \le c_\chi\|u\|_{H^1(\omega)}$ and $\|\zeta\|^2_{L^2(\omega)} \le c_\zeta\|u\|_{H^1(\omega)}$. Together with the continuity of the trace operator in $H^1(\omega)$

$$\|(u,\theta)\|_{H^1(\partial\omega_1)} \le c_1\|(u,\theta)\|_{H^1(\omega)}$$

this yields

$$\begin{aligned}
\big|J((C,D),(u,\theta))\big| &\le \frac{t}{2}\|C\|_{L^\infty(\omega)}c_\gamma\|u\|_{H^1(\omega)} + \frac{t^3}{24}\|C\|_{L^\infty(\omega)}c_\chi\|(u,\theta)\|_{H^1(\omega)} \\
&\quad + \frac{t}{2}\|D\|_{L^\infty(\omega)}c_\zeta\|(u,\theta)\|_{H^1(\omega)} + t\|f\|_{L^2(\omega)}\|u\|_{H^1(\omega)} \\
&\quad + \|g\|_{L^2(\partial\omega_1)}\|u\|_{H^1(\omega)} + \|h\|_{L^2(\partial\omega_1)}\|\theta\|_{H^1(\omega)}\,.
\end{aligned}$$

This yields the continuity required in (i) and (ii). As the linearity of the mapping $(C,D) \mapsto J((C,D),(u,\theta))$ implies convexity, (ii) is valid. The concavity required in (i) follows from the linear dependence of the strains on the displacements, the quadratic dependence of J on the strains and from the positive semidefiniteness of C and D.

Finally we show (iii): In [13] the ellipticity of Naghdi's shell model with nonhomogeneous anisotropic material has been proven using the assumption $\Phi \in C^3(\omega)$. In [5] it was shown that this assumption can be weakened to $\Phi \in W^{2,\infty}$ in the case of isotropic materials. Consequently it suffices to choose isotropic matrices

$$C_0 = \begin{pmatrix} 2\mu + \frac{2\lambda\mu}{\lambda+2\mu} & \frac{2\lambda\mu}{\lambda+2\mu} & 0 \\ \frac{2\lambda\mu}{\lambda+2\mu} & 2\mu + \frac{2\lambda\mu}{\lambda+2\mu} & 0 \\ 0 & 0 & 2\mu \end{pmatrix} \quad \text{and} \quad D_0 = \begin{pmatrix} 4\mu & 0 \\ 0 & 4\mu \end{pmatrix} \tag{24}$$

fulfilling the constraints on the traces given in (19) and (20) in order to show that (iii) is valid. As the summed trace is given by

$$t\,\mathrm{tr}(C) + \frac{1}{2}t\,\mathrm{tr}(D) = t\left(10\mu + \frac{4\lambda\mu}{\lambda+2\mu}\right) = t\left(14\mu - \frac{8\mu^2}{\lambda+2\mu}\right),$$

a possible choice is

$$\lambda \in \mathbb{R}^+ \quad , \quad \mu \le \min\left\{\frac{V}{14t|\omega|}, \frac{\rho^+}{14t}\right\}. \tag{25}$$

This proves the existence of a saddle-point of the functional $J((C,D),(u,\theta))$. $\square$

4 The Primal Problem

In [26] it has been shown that the Free Material Optimization problem for solid material can be transformed into a linear quadratically constrained optimization problem using duality theory. During this section we show that a similar technique can be applied on the Free Material optimization problem for shells resulting in a convex nonlinear semidefinite programm instead of the saddle-point problem given in (22).

Theorem 2. *Problem (22) is equivalent to the Lagrange dual problem of*

$$\begin{array}{ll} \max\limits_{\substack{(u,\theta)\in\mathcal{U} \\ \alpha\in\mathbb{R}_0^+ \\ \beta_{u,l}\in L^1(\omega) \\ \beta_{u,l}\ge 0}} & \displaystyle\int_\omega tf\cdot u\,dS + \int_{\partial\omega_1}(g\cdot u + h\cdot\theta)dl - \alpha V - \rho^+\int_\omega \beta_u dS \\ \textit{subject to} & \frac{1}{2}t\gamma(u)\gamma(u)^\top + \frac{t^3}{24}\chi(u,\theta)\chi(u,\theta)^\top - t(\alpha+\beta_u-\beta_l)E_3 \preceq 0 \\ & \frac{1}{2}t\zeta(u,\theta)\zeta(u,\theta)^\top - \frac{1}{2}t(\alpha+\beta_u-\beta_l)E_2 \preceq 0 \end{array} \tag{26}$$

where E_n denotes the unit matrix in $\mathbb{R}^n$.

Proof. We start with constructing the Lagrange function of problem (26). As the problem has matrix constraints the Lagrange multipliers are positive semidefinite matrices:

$$\begin{aligned}\min_{\substack{(C,D)\in L^\infty\\ C=C^\top\succeq 0\\ D=D^\top\succeq 0}} \max_{\substack{(u,\theta)\in\mathcal{U}\\ \alpha\in\mathbb{R}_0^+\\ \beta_{u,l}\in L^1(\omega)\\ \beta_{u,l}\geq 0}} L((u,\theta),(C,D),\alpha,\beta) = &\int_\omega tf\cdot u\,dS + \int_{\partial\omega_1}(g\cdot u + h\cdot\theta)dl\\ &-\alpha V - \rho^+\int_\omega \beta_u dS\\ &-\int_\omega \langle C, \frac{1}{2}t\gamma\gamma^\top + \frac{t^3}{24}\chi\chi^\top - t\alpha E_3 - t\beta_u E_3 + t\beta_l E_3\rangle dS\\ &-\int_\omega \langle D, \frac{1}{2}t\zeta\zeta^\top - \frac{1}{2}t\alpha E_2 - \frac{1}{2}t\beta_u E_2 + \frac{1}{2}t\beta_l E_2\rangle dS\,.\end{aligned} \tag{27}$$

We now regroup the terms in the Lagrangian and use the fact that for example

$$\langle C, t\alpha E_3\rangle = \operatorname{tr}(C\cdot t\alpha E_3) = t\alpha\operatorname{tr}(C) \tag{28}$$

to obtain

$$\begin{aligned}\min_{\substack{(C,D)\in L^\infty\\ C=C^\top\succeq 0\\ D=D^\top\succeq 0}} \max_{\substack{(u,\theta)\in\mathcal{U}\\ \alpha\in\mathbb{R}_0^+\\ \beta_{u,l}\in L^1(\omega)\\ \beta_{u,l}\geq 0}} L((u,\theta),(C,D),\alpha,\beta) = &-\int_\omega \langle C, \frac{1}{2}t\gamma\gamma^\top\rangle dS - \int_\omega \langle C, \frac{t^3}{24}\chi\chi^\top\rangle dS\\ &-\int_\omega \langle D, \frac{1}{2}t\zeta\zeta^\top\rangle dS + \int_\omega tf\cdot u\,dS + \int_{\partial\omega_1}(g\cdot u + h\cdot\theta)dl\\ &+\alpha\int_\omega t\operatorname{tr}(C) + \frac{1}{2}t\operatorname{tr}(D)dS - \alpha V\\ &+\int_\omega \beta_u\left(t\operatorname{tr}(C) + \frac{1}{2}t\operatorname{tr}(D) - \rho^+\right)dS\\ &-\int_\omega \beta_l\left(t\operatorname{tr}(C) + \frac{1}{2}t\operatorname{tr}(D)\right)dS\,.\end{aligned} \tag{29}$$

Next we make use of the following property of the trace

$$\int_\omega \langle C, t\gamma\gamma^\top\rangle dS = \int_\omega \operatorname{tr}(C\cdot t\gamma\gamma^\top)dS = \int_\omega t\gamma^\top C\gamma\,dS \tag{30}$$

and get

$$\begin{aligned}\min_{\substack{(C,D)\in L^\infty\\ C=C^\top\succeq 0\\ D=D^\top\succeq 0}} \max_{\substack{(u,\theta)\in\mathcal{U}\\ \alpha\in\mathbb{R}_0^+\\ \beta_{u,l}\in L^1(\omega)\\ \beta_{u,l}\geq 0}} L((u,\theta),(C,D),\alpha,\beta) = &-\frac{1}{2}\int_\omega \Big[t\gamma^\top C\gamma + \frac{t^3}{12}\chi^\top C\chi\\ &+ t\zeta^\top D\zeta\Big]\,dS + \int_\omega tf\cdot u\,dS + \int_{\partial\omega_1}(g\cdot u + h\cdot\theta)dl\\ &+\alpha\int_\omega t\operatorname{tr}(C) + \frac{1}{2}t\operatorname{tr}(D)dS - \alpha V\\ &+\int_\omega \beta_u\left(t\operatorname{tr}(C) + \frac{1}{2}t\operatorname{tr}(D) - \rho^+\right)dS\\ &-\int_\omega \beta_l\left(t\operatorname{tr}(C) + \frac{1}{2}t\operatorname{tr}(D)\right)dS\,.\end{aligned} \tag{31}$$

It now remains to show that problem (31) is equivalent to problem (22) in the sense that each saddle-point $((C^*, D^*), (u^*, \theta^*))$ of (22) can be extended to a saddle-point $((C^*, D^*), (u^*, \theta^*, \alpha^*, {\beta_{l,u}}^*))$ of (31) and each saddle-point $((C^*, D^*), (u^*, \theta^*, \alpha^*, {\beta_{l,u}}^*))$ of (31) gives a saddle-point $((C^*, D^*), (u^*, \theta^*))$ of (22). The proof follows the ideas of [26, Theorem 3.3.1], however it has to be generalized to matrices. For the given saddle-point functionals of problem (22)

$$\begin{aligned} J_D((C,D),(u,\theta)) \quad &= \quad -\frac{1}{2}\int_\omega t\gamma^\top C\gamma + \frac{t^3}{12}\chi^\top C\chi + t\zeta^\top D\zeta \, dS + \int_\omega tf\cdot u\, dS \\ &\quad + \int_{\partial\omega_1} g\cdot u + h\cdot\theta\, dl \end{aligned} \tag{32}$$

and problem (31)

$$\begin{aligned} &J_L((C,D),(u,\theta,\alpha,\beta)) = \\ &= \; -\frac{1}{2}\int_\omega t\gamma^\top C\gamma + \frac{t^3}{12}\chi^\top C\chi + t\zeta^\top D\zeta \, dS + \int_\omega tf\cdot u\, dS \\ &\quad + \int_{\partial\omega_1} g\cdot u + h\cdot\theta\, dl + \alpha\int_\omega t\,\mathrm{tr}C + \frac{t}{2}\mathrm{tr}D\, dS - \alpha V \\ &\quad + \int_\omega \beta_u\left(t\,\mathrm{tr}C + \frac{t}{2}\mathrm{tr}D - \rho^+\right) dS - \int_\omega \beta_l\left(t\,\mathrm{tr}C + \frac{t}{2}\mathrm{tr}D\right) dS \end{aligned} \tag{33}$$

the Gâteaux derivatives can be computed as

$$\delta_{C,D}J_D((C,D),(u,\theta))(\delta C,\delta D) \quad = \quad -\frac{1}{2}\int_\omega t\gamma^\top\delta C\gamma + \frac{t^3}{12}\chi^\top\delta C\chi + t\zeta^\top\delta D\zeta\, dS \tag{34}$$

$$\begin{aligned} \delta_{u,\theta}J_D((C,D),(u,\theta))(\delta u,\delta\theta) \quad &= \quad -\int_\omega t\gamma^\top(u)C\gamma(\delta u) + \frac{t^3}{12}\chi^\top(u,\theta)C\chi(\delta u,\delta\theta) \\ &\quad + t\zeta^\top(u,\theta)D\zeta(\delta u,\delta\theta)\, dS + \int_\omega tf\cdot\delta u\, dS \\ &\quad + \int_{\partial\omega_1} g\cdot\delta u + h\cdot\delta\theta\, dl \end{aligned} \tag{35}$$

$$\begin{aligned} \delta_{C,D}J_L((C,D),(u,\theta,\alpha,\beta))(\delta C,\delta D) \quad &= \quad -\frac{1}{2}\int_\omega t\gamma^\top\delta C\gamma + \frac{t^3}{12}\chi^\top\delta C\chi + t\zeta^\top\delta D\zeta\, dS \\ &\quad + \alpha\int_\omega t\,\mathrm{tr}\delta C + \frac{t}{2}\mathrm{tr}\delta D\, dS \\ &\quad + \int_\omega \beta_u\left(t\,\mathrm{tr}\delta C + \frac{t}{2}\mathrm{tr}\delta D\right) dS \\ &\quad - \int_\omega \beta_l\left(t\,\mathrm{tr}\delta C + \frac{t}{2}\mathrm{tr}\delta D\right) dS \end{aligned} \tag{36}$$

and

$$
\begin{aligned}
&\delta_{u,\theta,\alpha,\beta} J_L((C,D),(u,\theta,\alpha,\beta))(\delta u,\delta\theta,\delta\alpha,\delta\beta) = \\
&\quad -\int_\omega t\gamma^\top(u)C\gamma(\delta u) + \frac{t^3}{12}\chi^\top(u,\theta)C\chi(\delta u,\delta\theta) + t\zeta^\top(u,\theta)D\zeta(\delta u,\delta\theta)\,dS \\
&\quad + \int_\omega tf\cdot\delta u\,dS + \int_{\partial\omega_1} g\cdot\delta u + h\cdot\delta\theta\,dl + \delta\alpha\left(\int_\omega t\,\mathrm{tr}C + \frac{t}{2}\,\mathrm{tr}D\,dS - V\right) \\
&\quad + \int_\omega \delta\beta_u\left(t\,\mathrm{tr}C + \frac{t}{2}\,\mathrm{tr}D - \rho^+\right)dS - \int_\omega \delta\beta_l\left(t\,\mathrm{tr}C + \frac{t}{2}\,\mathrm{tr}D\right)dS\,, \qquad (37)
\end{aligned}
$$

respectively. To characterize saddle-points we define

$$
\begin{aligned}
\tilde{\mathcal{C}} &:= \{(C,D)\in[L^\infty(\omega)]^{3\times 3}\times[L^\infty(\omega)]^{2\times 2}\,|\,C=C^\top\succeq 0\,,\ D=D^\top\succeq 0\}\,, && (38)\\
\mathcal{L} &:= \{(\alpha,\beta)\in\mathbb{R}_0^+\times[L^1(\omega)]^2\,|\,\beta\geq 0\} && (39)
\end{aligned}
$$

and use the following Lemma, whose proof is given in appendix A:

Lemma 1. *We consider the saddle-point problem*

$$
\min_{p\in\mathcal{A}}\max_{u\in\mathcal{B}} J(p,u) \qquad (40)
$$

where $\mathcal{A}=\mathcal{C}$, $\mathcal{B}=\mathcal{U}$ and $J=J_D$. Then $(p^*,u^*)\in\mathcal{A}\times\mathcal{B}$ is a saddle-point of J if and only if

$$
\begin{aligned}
\delta_p J(p^*,u^*)(p-p^*) &\geq 0 \quad \forall\, p\in\mathcal{A}\,,\\
\delta_u J(p^*,u^*)(u-u^*) &\leq 0 \quad \forall\, u\in\mathcal{B}\,. \qquad (41)
\end{aligned}
$$

The same is true for $\mathcal{A}=\tilde{\mathcal{C}}$, $\mathcal{B}=\mathcal{U}\times\mathcal{L}$ and $J=J_L$.

Thus the saddle-point conditions for problem (22) can be given as

$$
\begin{aligned}
&\delta_{C,D} J_D((C^*,D^*),(u^*,\theta^*))(C-C^*,D-D^*) = \qquad (42)\\
&= -\frac{1}{2}\int_\omega t\gamma^\top(C-C^*)\gamma + \frac{t^3}{12}\chi^\top(C-C^*)\chi + t\zeta^\top(D-D^*)\zeta\,dS \geq 0 \quad \forall\,(C,D)\in\mathcal{C}\,,
\end{aligned}
$$

$$
\begin{aligned}
&\delta_{u,\theta} J_D((C^*,D^*),(u^*,\theta^*))(u-u^*,\theta-\theta^*) = \qquad (43)\\
&= -\int_\omega t\gamma^\top(u^*)C^*\gamma(u-u^*) + \frac{t^3}{12}\chi^\top(u^*,\theta^*)C^*\chi(u-u^*,\theta-\theta^*)\\
&\quad + t\zeta^\top(u^*,\theta^*)D^*\zeta(u-u^*,\theta-\theta^*)\,dS\\
&\quad + \int_\omega tf\cdot(u-u^*)\,dS + \int_{\partial\omega_1} g\cdot(u-u^*) + h\cdot(\theta-\theta^*)\,dl \leq 0 \quad \forall\,(u,\theta)\in\mathcal{U}
\end{aligned}
$$

and on the other hand $((C^*, D^*), (u^*, \theta^*, \alpha^*, \beta^*))$ is a saddle-point of (31) if and only if

$$\begin{aligned}
&\delta_{C,D} J_L((C^*, D^*), (u^*, \theta^*, \alpha^*, \beta^*))(C - C^*, D - D^*) = \qquad (44)\\
&= \quad -\frac{1}{2}\int_\omega t\gamma^\top (C - C^*)\gamma + \frac{t^3}{12}\chi^\top (C - C^*)\chi + t\zeta^\top (D - D^*)\zeta \, dS \\
&\quad + \alpha^* \int_\omega t \,\mathrm{tr}(C - C^*) + \frac{t}{2}\,\mathrm{tr}(D - D^*)\, dS \\
&\quad + \int_\omega \beta_u{}^* \left(t\,\mathrm{tr}(C - C^*) + \frac{t}{2}\,\mathrm{tr}(D - D^*)\right) dS \\
&\quad - \int_\omega \beta_l{}^* \left(t\,\mathrm{tr}(C - C^*) + \frac{t}{2}\,\mathrm{tr}(D - D^*)\right) dS \geq 0 \quad \forall\, (C, D) \in \tilde{\mathcal{C}}
\end{aligned}$$

and

$$\begin{aligned}
&\delta_{u,\theta,\alpha,\beta} J_L((C^*, D^*), (u^*, \theta^*, \alpha^*, \beta^*))(u - u^*, \theta - \theta^*, \alpha - \alpha^*, \beta - \beta^*) = \qquad (45)\\
&\quad -\int_\omega t\gamma^\top (u^*) C^* \gamma(u - u^*) + \frac{t^3}{12}\chi^\top (u^*, \theta^*) C^* \chi(u - u^*, \theta - \theta^*) \\
&\quad + t\zeta^\top (u^*, \theta^*) D^* \zeta(u - u^*, \theta - \theta^*)\, dS + \int_\omega tf \cdot (u - u^*)\, dS \\
&\quad + \int_{\partial\omega_1} g \cdot (u - u^*) + h \cdot (\theta - \theta^*)\, dl + (\alpha - \alpha^*)\left(\int_\omega t\,\mathrm{tr} C^* + \frac{t}{2}\,\mathrm{tr} D^*\, dS - V \right) \\
&\quad + \int_\omega (\beta_u - \beta_u{}^*)\left(t\,\mathrm{tr} C^* + \frac{t}{2}\,\mathrm{tr} D^* - \rho^+ \right) dS - \int_\omega (\beta_l - \beta_l{}^*)\left(t\,\mathrm{tr} C^* + \frac{t}{2}\,\mathrm{tr} D^* \right) dS \\
&\leq \quad 0 \quad \forall\, (u, \theta, \alpha, \beta) \in \mathcal{U} \times \mathcal{L}\,.
\end{aligned}$$

Now we first show, that (44) and (45) yield (42) and (43), thus each saddle-point $((C^*, D^*), (u^*, \theta^*, \alpha^*, \beta^*))$ of problem (31) gives a saddle-point $((C^*, D^*), (u^*, \theta^*))$ of (22). Setting $\alpha = \alpha^*$ and $\beta = \beta^*$ in (45) yields (43). Furthermore positivity of α and β together with the existence of the saddle-point $((C^*, D^*), (u^*, \theta^*, \alpha^*, \beta^*))$ (and thus existence of a finite solution) gives

$$\begin{aligned}
\int_\omega t\,\mathrm{tr} C^* + \frac{t}{2}\,\mathrm{tr} D^*\, dS \quad &\leq \quad V\,, \\
0 \leq t\,\mathrm{tr} C^* + \frac{t}{2}\,\mathrm{tr} D^* \quad &\leq \quad \rho^+ \qquad \text{a. e. in } \omega\,.
\end{aligned}$$

Thus $(C^*, D^*) \in \mathcal{C}$. Two cases can be distinguished: $\alpha^* = 0$ or $\alpha^* > 0$. In the latter case take a look at (45) for $u = u^*$, $\theta = \theta^*$ and $\beta = \beta^*$:

$$(\alpha - \alpha^*) \underbrace{\left(\int_\omega t\,\mathrm{tr} C^* + \frac{t}{2}\,\mathrm{tr} D^*\, dS - V \right)}_{\leq 0} \leq 0 \quad \forall\, \alpha \in \mathbb{R}_0^+\,.$$

While this is always true for $\alpha^* = 0$, in the case $\alpha^* > 0$ we can choose $\alpha = \frac{\alpha^*}{2} \in \mathbb{R}^+$ and get

$$\underbrace{-\frac{\alpha^*}{2}}_{<0} \underbrace{\left(\int_\omega t \operatorname{tr} C^* + \frac{t}{2} \operatorname{tr} D^* \, dS - V \right)}_{\leq 0} \leq 0 \,.$$

Obviously the only way to fulfill this inequality in the case $\alpha^* > 0$ for all $\alpha \in \mathbb{R}_0^+$ is

$$\int_\omega t \operatorname{tr} C^* + \frac{t}{2} \operatorname{tr} D^* \, dS = V \,. \tag{46}$$

The same idea works for β. Set all variables in (45) equal to their optimal values except β_u. This yields

$$\int_\omega (\beta_u - {\beta_u}^*) \left(t \operatorname{tr} C^* + \frac{t}{2} \operatorname{tr} D^* - \rho^+ \right) \, dS \leq 0 \quad \forall \, \beta_u \in L^1(\omega), \beta_u \geq 0 \,.$$

If there exists a subset $\tilde{\omega} \subset \omega$, $|\tilde{\omega}| > 0$ with ${\beta_u}^*(x) > 0$ a. e. in $\tilde{\omega}$, set

$$\beta_u = \begin{cases} {\beta_u}^* & \text{in } \omega \setminus \tilde{\omega} \\ \frac{{\beta_u}^*}{2} & \text{in } \tilde{\omega} \end{cases}$$

$$\begin{aligned} \Rightarrow \quad \int_\omega (\beta_u - {\beta_u}^*) \left(t \operatorname{tr} C^* + \frac{t}{2} \operatorname{tr} D^* - \rho^+ \right) \, dS \quad &= \\ \int_{\tilde{\omega}} \underbrace{-\frac{{\beta_u}^*}{2}}_{<0} \underbrace{\left(t \operatorname{tr} C^* + \frac{t}{2} \operatorname{tr} D^* - \rho^+ \right)}_{\leq 0} \, dS \quad &\leq \quad 0 \,. \end{aligned}$$

In the case ${\beta_u}^* > 0$ this can only be true for

$$t \operatorname{tr} C^* + \frac{t}{2} \operatorname{tr} D^* = \rho^+ \qquad \text{a.e. in } \tilde{\omega} \,. \tag{47}$$

Analogously it follows that for each subset $\tilde{\omega} \subset \omega$, $|\tilde{\omega}| > 0$ with ${\beta_l}^* > 0$ a.e. in $\tilde{\omega}$ this results in

$$t \operatorname{tr} C^* + \frac{t}{2} \operatorname{tr} D^* = 0 \qquad \text{a.e. in } \tilde{\omega} \,. \tag{48}$$

It remains to show (42). Comparison with (44) shows that it suffices to show

$$\begin{aligned} \alpha^* \int_\omega t \operatorname{tr}(C - C^*) + \frac{t}{2} \operatorname{tr}(D - D^*) \, dS + \int_\omega {\beta_u}^* \left(t \operatorname{tr}(C - C^*) + \frac{t}{2} \operatorname{tr}(D - D^*) \right) \, dS \\ - \int_\omega {\beta_l}^* \left(t \operatorname{tr}(C - C^*) + \frac{t}{2} \operatorname{tr}(D - D^*) \right) \, dS \leq 0 \quad \forall \, (C, D) \in \mathcal{C} \,. \end{aligned} \tag{49}$$

The last two terms are non-positive due to (47) and (48), while the first term is non-positive due to (46). This shows (42) and thus this direction of the proof is completed.

The last part of the proof consists of showing that each saddle-point $((C^*, D^*), (u^*, \theta^*))$ of problem (22) can be extended to a saddle-point $((C^*, D^*), (u^*, \theta^*, \alpha^*, \beta^*))$ of (31) or, in other words, (42) and (43) give (44) and (45). To this end we first show a local maximum principle for matrices.

Lemma 2. *If the maximum of $\int_\omega {s_1}^\top M s_1 + {s_2}^\top M s_2 + \ldots + {s_p}^\top M s_p \, dS$ is attained at $M = M^*$ (for $M \in [L^\infty(\omega)]^{n\times n}$, $M = M^\top \succeq 0$, $\rho^- \leq \mathrm{tr}M \leq \rho^+$, $\int_\omega \mathrm{tr}M \, dS \leq V$ and $s_i \in \left[L^2(\omega)\right]^n$, $i = 1, \ldots, p$), then there exists a Lagrange multiplier $\alpha^* \in \mathbb{R}$ such that the following statements hold:*

(i)

$$\int_\omega \langle M - M^*, -\sum_{i=1}^{p} s_i {s_i}^\top + \alpha^* E_n \rangle dS \leq 0 \tag{50}$$

for all $M \in [L^\infty(\omega)]^{n\times n}$, $M = M^\top \succeq 0$ and $\rho^- \leq \mathrm{tr}M \leq \rho^+$.

(ii) Define by $\sum_{j=1}^n \lambda_j e_j {e_j}^\top$ the eigenvalue decomposition of $\sum_{i=1}^p s_i {s_i}^\top$ with ${e_j}^\top e_k = \delta_{jk}$ and $\|e_j\|_{L^2} = 1 \; \forall j = 1, \ldots, n$. For each subset $\tilde{\omega} \subset \omega$, $|\tilde{\omega}| > 0$ with $\mathrm{tr}M^* > 0$ a. e. there exists at least one $e_j \notin \ker(M^*)$ whose associated eigenvalue $\lambda_j > 0$.

(iii) For each subset $\tilde{\omega} \subset \omega$, $|\tilde{\omega}| > 0$ with

$$\max_{\substack{j=1,\ldots,n \\ e_j \notin \ker(M^*)}} \lambda_j < \alpha^* \quad \Rightarrow \quad \mathrm{tr}M^*(x) = \rho^- \; \text{a. e. in } \tilde{\omega} \,. \tag{51}$$

(iv) For each subset $\tilde{\omega} \subset \omega$, $|\tilde{\omega}| > 0$ with

$$\min_{\substack{j=1,\ldots,n \\ e_j \notin \ker(M^*)}} \lambda_j > \alpha^* \quad \Rightarrow \quad \mathrm{tr}M^*(x) = \rho^+ \; \text{a. e. in } \tilde{\omega} \,. \tag{52}$$

The proof is given in appendix A. For a given saddle-point $((C^*, D^*), (u^*, \theta^*))$ of (22) we now define

$$s_1 = \begin{pmatrix} \frac{1}{\sqrt{2}}\gamma(u^*) \\ 0 \\ 0 \end{pmatrix}, \; s_2 = \begin{pmatrix} \frac{t}{\sqrt{24}}\chi(u^*, \theta^*) \\ 0 \\ 0 \end{pmatrix}, \; s_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \zeta(u^*, \theta^*) \end{pmatrix} \tag{53}$$

and the matrix

$$M = \begin{pmatrix} t\,C & 0 \\ 0 & \frac{1}{2} t\, D \end{pmatrix}. \tag{54}$$

Thus in the present case we have $p = 3$ and $n = 5$. We also define

$$\mathcal{J} := \{ j \in \{1, \ldots n\} \,|\, e_j \notin \ker(M^*) \}\,. \tag{55}$$

Then according to Lemma 2 there exists an $\alpha^* \in \mathbb{R}$ such that for each $\tilde{\omega} \subset \omega$, $|\tilde{\omega}| > 0$ with

$$\begin{aligned} \max_{j\in\mathcal{J}} \lambda_j < \alpha^* \quad &\Rightarrow \quad \mathrm{tr} M^* = \rho^- \text{ a.e. in } \tilde{\omega}\,, \\ \min_{j\in\mathcal{J}} \lambda_j > \alpha^* \quad &\Rightarrow \quad \mathrm{tr} M^* = \rho^+ \text{ a.e. in } \tilde{\omega}\,. \end{aligned}$$

Define

$$\beta_u{}^* = \begin{cases} \max\limits_{j=1,\dots,n} \lambda_j - \alpha^* & \text{if } \min\limits_{j\in\mathcal{J}} \lambda_j > \alpha^* \\ 0 & \text{else} \end{cases} \tag{56}$$

$$\beta_l{}^* = \begin{cases} \alpha^* - \max\limits_{j=1,\dots,n} \lambda_j & \text{if } \max\limits_{j\in\mathcal{J}} \lambda_j < \alpha^* \text{ or } \nexists\, e_j \notin \ker(M^*) \\ 0 & \text{else} \end{cases} \tag{57}$$

Consider now the case $\alpha^* > 0$. Suppose $\int_\omega \mathrm{tr} M^* \, dS < V$ and there exists $\tilde{\omega} \subset \omega$, $|\tilde{\omega}| > 0$ with $\rho^- < \mathrm{tr} M^* < \rho^+$ a.e. on $\tilde{\omega}$ (which is true for a sensible choice of V, ρ^- and ρ^+). Due to Lemma 2 we know there exists a $\lambda_k > 0$ with the associated eigenfunction $e_k \notin \ker(M^*)$. Define the matrix

$$\tilde{M} = M^* + \chi_{\tilde{\omega}} e_k e_k{}^\top \frac{1}{|\tilde{\omega}|} \min\left\{ \left(V - \int_\omega \mathrm{tr} M^* \, dS\right), \left(\rho^+ - \mathrm{tr} M^*\right) \right\} ,$$

where $\chi_{\tilde{\omega}}$ is the characteristic function of the set $\tilde{\omega}$:

$$\chi_{\tilde{\omega}}(x) = \begin{cases} 1 & \text{if } x \in \tilde{\omega} \\ 0 & \text{else} \end{cases} \,. \tag{58}$$

Obviously we have $\tilde{M} = \tilde{M}^\top \succeq 0$, $\int_\omega \mathrm{tr} \tilde{M} \, dS \le V$ and $0 \le \rho^- \le \mathrm{tr} \tilde{M} \le \rho^+$ a.e. on ω. Thus $\tilde{M}$ is contained in the admissible set. Now take a look at the objective

$$\begin{aligned} -\int_\omega \langle \tilde{M}, \sum_{j=1}^n \lambda_j e_j e_j{}^\top \rangle \, dS \;&=\; -\int_\omega \langle M^*, \sum_{i=1}^p s_i s_i{}^\top \rangle \, dS \\ &\quad\; - \int_{\tilde{\omega}} \underbrace{\frac{1}{|\tilde{\omega}|}}_{>0} \underbrace{\min\left\{ \left(V - \int_\omega \mathrm{tr} M^* \, dS\right), \left(\rho^+ - \mathrm{tr} M^*\right) \right\}}_{>0} \underbrace{\lambda_k}_{>0} \, dS \\ &<\; -\int_\omega \langle M^*, \sum_{i=1}^p s_i s_i{}^\top \rangle \, dS\,. \end{aligned}$$

This contradicts the optimality of M^*. Thus for $\alpha^* > 0$ we must have $\int_\omega \mathrm{tr} M^* \, dS = V$. The case $\alpha^* < 0$ is not taken into consideration as $\min_{j=1,\dots,n} \lambda_j \ge 0$, therefore $\alpha^* < 0$ would result in $\mathrm{tr} M^* = \rho^+$ a. e. on ω. With the above definition of $\beta_u{}^*$ and $\beta_l{}^*$ it follows that

$$\begin{aligned} (\alpha - \alpha^*) \left(\int_\omega \mathrm{tr} M^* \, dS - V \right) &+ \int_\omega (\beta_u - \beta_u{}^*) \left(\mathrm{tr} M^* - \rho^+\right) \, dS \\ - \int_\omega (\beta_l - \beta_l{}^*) \mathrm{tr} M^* \, dS \le 0 \qquad &\forall\, \alpha \in \mathbb{R}_0^+ \,,\, \beta \in \left[L^1(\omega)\right]^2 \,,\, \beta \ge 0\,. \end{aligned} \tag{59}$$

Together with (43) this yields (45). To show (44) we again refer to Lemma 2. According to (50) we have

$$\int_\omega \langle M - M^*, -\sum_{i=1}^{p} s_i {s_i}^\top + \alpha^* E_n \rangle \, dS \geq 0 \quad \forall\, M \in \mathcal{M} \, .$$

We want to show (44)

$$\int_\omega \langle M - M^*, -\sum_{i=1}^{p} s_i {s_i}^\top + (\alpha^* + {\beta_u}^* - {\beta_l}^*) E_n \rangle dS \geq 0$$

for all $M \in [L^\infty(\omega)]^{n\times n}$, $M = M^\top \succeq 0$. It is sufficient to prove this statement for all subsets $\tilde{\omega} \subset \omega$, $|\tilde{\omega}| > 0$. Here we distinguish three different cases:

1. $\mathrm{tr} M^* = \rho^+$ a.e. on $\tilde{\omega}$.

   According to Lemma 2 (ii) there exists at least one eigenfunction e_k such that ${e_k}^\top M^* e_k = \mu_k > 0$. The value of the trace is independent from the chosen coordinate system thus we have

   $$\sum_{j=1}^{n} {e_j}^\top M^* e_j = \mathrm{tr} M^* = \rho^+ = \sum_{j \in \mathcal{J}} {e_j}^\top M^* e_j \, . \tag{60}$$

   If there exists only one eigenfunction $e_k \notin \ker(M^*)$ this yields $\mu_k = \rho^+$ and for M^* it holds that $M^* = \rho^+ v v^\top$ with $\|v\| = 1$. Due to $\rho^+ = {e_k}^\top M^* e_k = {e_k}^\top \rho^+ v v^\top e_k = \rho^+ \left(v^\top e_k\right)^2$ we get $v^\top e_k = \pm 1$ and as $\sum_{j=1}^{n} \left(v^\top e_j\right)^2 = 1$ it follows that $v^\top e_j = 0 \; \forall\, j \neq k$ and thus $v = \pm e_k$ resulting in $M^* = \rho^+ e_k {e_k}^\top$.

   If there exist more than one eigenfunction not in the kernel of M^*, for example

   $$0 < {e_k}^\top M^* e_k = \mu_k < \rho^+ \quad , \quad 0 < {e_l}^\top M^* e_l = \mu_l \leq \rho^+ - \mu_k \, ,$$

   we can assume without loss of generality that $\lambda_k \geq \lambda_l$ holds. Consider the matrix

   $$\tilde{M} = M^* - \mu_l e_l {e_l}^\top + \mu_l e_k {e_k}^\top \in \mathcal{M} \, .$$

   Inserting this into (50) yields

   $$\begin{aligned} \int_{\tilde{\omega}} \langle \tilde{M} - M^*, \sum_{j=1}^{n} (-\lambda_j + \alpha^*) e_j {e_j}^\top \rangle \, dS &= \int_{\tilde{\omega}} \langle -\mu_l e_l {e_l}^\top + \mu_l e_k {e_k}^\top, \sum_{j=1}^{n} (-\lambda_j + \alpha^*) e_j {e_j}^\top \rangle \, dS \\ &= \int_{\tilde{\omega}} -\mu_l(-\lambda_l + \alpha^*) + \mu_l(-\lambda_k + \alpha^*) \, dS \\ &= \int_{\tilde{\omega}} \underbrace{\mu_l}_{>0} \underbrace{(\lambda_l - \lambda_k)}_{\leq 0} \, dS \geq 0 \, . \end{aligned}$$

This can only be true for $\lambda_k = \lambda_l =: \tilde{\lambda}$. As such a matrix $\tilde{M}$ can be constructedfor any pair of eigenfunctions $e_j, e_k \notin \ker(M^*)$, this results in $\lambda_j = \tilde{\lambda}\ \forall j \in \mathcal{J}$. Suppose this $\tilde{\lambda}$ would not be the maximal eigenvalue, in other words:

$$\exists \lambda_m = \max_{j=1,\dots,n} \lambda_j > \tilde{\lambda}\,.$$

Then set $M_m = \rho^+ e_m {e_m}^\top \in \mathcal{M}$ and inserting this into (50)

$$\begin{aligned}
&\int_{\tilde{\omega}} \langle M_m - M^*, \sum_{j=1}^n (-\lambda_j + \alpha^*) e_j {e_j}^\top \rangle \, dS = \\
&= \int_{\tilde{\omega}} (-\lambda_m + \alpha^*) \underbrace{{e_m}^\top M_m e_m}_{=\rho^+} - (-\tilde{\lambda} + \alpha^*) \underbrace{\sum_{j \in \mathcal{J}} {e_j}^\top M^* e_j}_{=\rho^+} \, dS \\
&= \int_{\tilde{\omega}} \rho^+ (\tilde{\lambda} - \lambda_m) \, dS < 0
\end{aligned}$$

contradicts the optimality of M^*. Thus we get

$$\tilde{\lambda} = \max_{j=1,\dots,n} \lambda_j = \max_{j \in \mathcal{J}} \lambda_j\,.$$

We now distinguish two cases. Either we have $\min_{j\in\mathcal{J}} \lambda_j > \alpha^*$ which results in ${\beta_u}^* > 0$ and ${\beta_l}^* = 0$ a.e. on $\tilde{\omega}$. Then we get

$$\begin{aligned}
&\int_{\tilde{\omega}} \langle M - M^*, \sum_{j=1}^n (-\lambda_j + \alpha^* + \underbrace{{\beta_u}^*}_{=\tilde{\lambda}-\alpha^*}) e_j {e_j}^\top \rangle \, dS = \\
&= \int_{\tilde{\omega}} \sum_{j=1}^n \underbrace{(\tilde{\lambda} - \lambda_j)}_{\geq 0} \underbrace{{e_j}^\top M e_j}_{\geq 0} - \sum_{j\in\mathcal{J}} \underbrace{(\tilde{\lambda} - \tilde{\lambda})}_{=0} {e_j}^\top M^* e_j \, dS \\
&\geq 0 \quad \forall M \in \tilde{C}\,.
\end{aligned}$$

The other possibility is $\min_{j\in\mathcal{J}} \lambda_j \leq \alpha^* \leq \max_{j\in\mathcal{J}} \lambda_j$ and thus ${\beta_u}^* = {\beta_l}^* = 0$ a.e. on $\tilde{\omega}$. But as $\min_{j\in\mathcal{J}} \lambda_j = \max_{j\in\mathcal{J}} \lambda_j = \tilde{\lambda}$ this yields $\tilde{\lambda} = \alpha^*$ and results in

$$\begin{aligned}
&\int_{\tilde{\omega}} \langle M - M^*, \sum_{j=1}^n (-\lambda_j + \alpha^*) e_j {e_j}^\top \rangle \, dS = \\
&= \int_{\tilde{\omega}} \sum_{j=1}^n \underbrace{(-\lambda_j + \alpha^*)}_{=-\lambda_j+\tilde{\lambda}\geq 0} \underbrace{{e_j}^\top M e_j}_{\geq 0} - \sum_{j\in\mathcal{J}} \underbrace{(-\tilde{\lambda} + \alpha^*)}_{=0} {e_j}^\top M^* e_j \, dS \\
&\geq 0 \quad \forall M \in \tilde{C}\,.
\end{aligned}$$

2. $\mathrm{tr} M^* = \rho^-$ a.e. on $\tilde{\omega}$.

First assume $\rho^- > 0$. In this case we either have $\lambda_j = 0 \; \forall\, j = 1, \ldots, n$, then unless $\alpha^* = 0$ (which results in $\int_{\tilde{\omega}} \langle M - M^*, \sum_{j=1}^{n} (-\lambda_j + \alpha^*) e_j {e_j}^\top \rangle \, dS = 0$) we have ${\beta_l}^* = \alpha^* - \max_{j=1,\ldots,n} = \alpha^* - 0 = \alpha^*$ and thus

$$\int_{\tilde{\omega}} \langle M - M^*, \sum_{j=1}^{n} \underbrace{(-\lambda_j + \alpha^* - {\beta_l}^*)}_{=0} e_j {e_j}^\top \rangle \, dS = 0 \quad \forall\, M \in \tilde{C}\,. \tag{61}$$

On the other hand there might exist $\lambda_j > 0$, then it follows as in Lemma 2 (ii) that at least one eigenfunction $e_j \notin \ker(M^*)$. If only one eigenfunction $e_k \notin \ker(M^*)$ exists we get again (compare to 1.) that $M^* = \rho^- e_k {e_k}^\top$. Analogously we also derive that for more than one eigenfunction $e_j \notin \ker(M^*)$ the associated eigenvalues have to be equal and are the maximal eigenvalue of $\sum_{i=1}^{p} s_i {s_i}^\top$:

$$\forall\, j \in \mathcal{J} : \quad \lambda_j = \max_{j=1,\ldots,n} \lambda_j = \tilde{\lambda}\,.$$

Again two cases are distinguished. When $\max_{j \in \mathcal{J}} \lambda_j < \alpha^*$ it follows that ${\beta_u}^* = 0$, ${\beta_l}^* = \alpha^* - \tilde{\lambda}$ and this results in

$$\begin{aligned}
&\int_{\tilde{\omega}} \langle M - M^*, \sum_{j=1}^{n} (-\lambda_j + \alpha^* - {\beta_l}^*) e_j {e_j}^\top \rangle \, dS = \\
= \;& \int_{\tilde{\omega}} \sum_{j=1}^{n} \underbrace{(-\lambda_j + \tilde{\lambda})}_{\geq 0} \underbrace{{e_j}^\top M e_j}_{\geq 0} - \sum_{j \in \mathcal{J}} \underbrace{(-\tilde{\lambda} + \tilde{\lambda})}_{=0} {e_j}^\top M^* e_j \, dS \\
\geq \;& 0 \quad \forall\, M \in \tilde{\mathcal{C}}\,.
\end{aligned}$$

The other case is $\min_{j \in \mathcal{J}} \lambda_j \leq \alpha^* \leq \max_{j \in \mathcal{J}} \lambda_j$ resulting in ${\beta_u}^* = {\beta_l}^* = 0$. But due to $\min_{j \in \mathcal{J}} \lambda_j = \max_{j \in \mathcal{J}} \lambda_j = \tilde{\lambda}$ it holds that $\tilde{\lambda} = \alpha^*$ and thus

$$\begin{aligned}
&\int_{\tilde{\omega}} \langle M - M^*, \sum_{j=1}^{n} (-\lambda_j + \alpha^*) e_j {e_j}^\top \rangle \, dS = \\
= \;& \int_{\tilde{\omega}} \sum_{j=1}^{n} \underbrace{(-\lambda_j + \tilde{\lambda})}_{\geq 0} \underbrace{{e_j}^\top M e_j}_{\geq 0} - \sum_{j \in \mathcal{J}} \underbrace{(-\tilde{\lambda} + \tilde{\lambda})}_{=0} {e_j}^\top M^* e_j \, dS \\
\geq \;& 0 \quad \forall\, M \in \tilde{\mathcal{C}}\,.
\end{aligned}$$

In the special case $\rho^- = 0$ it holds that $M^* = 0$. Thus there is no $e_j \notin \ker(M^*)$. This results in ${\beta_u}^* = 0$ and ${\beta_l}^* = \alpha^* - \max_{j=1,\ldots,n} \lambda_j = \alpha^* - \tilde{\lambda}$. This yields

$$\int_{\tilde{\omega}} \langle \underbrace{M}_{\succeq 0} - 0, \sum_{j=1}^{n} \underbrace{(-\lambda_j + \alpha^* - \alpha^* + \tilde{\lambda})}_{\geq 0} e_j {e_j}^\top \rangle \, dS \geq 0 \quad \forall\, M \in \tilde{\mathcal{C}}\,.$$

3. $\mathrm{tr}M^* \in (\rho^-, \rho^+)$

   This results in $\beta_u{}^* = \beta_l{}^* = 0$ and due to (50) we already know

$$\int_{\tilde{\omega}} \langle M - M^*, \sum_{j=1}^{n} (-\lambda_j + \alpha^*) e_j {e_j}^\top \rangle \, dS \geq 0 \quad \forall\, M \in \mathcal{M}\,.$$

   Assume there exists a $\hat{M} \in \tilde{\mathcal{C}} \setminus \mathcal{M}$ with $\mathrm{tr}\hat{M} > \rho^+$ such that

$$\int_{\tilde{\omega}} \langle \hat{M} - M^*, \sum_{j=1}^{n} (-\lambda_j + \alpha^*) e_j {e_j}^\top \rangle \, dS < 0\,.$$

   Consider the matrix

$$\tilde{M} = M^* + (\hat{M} - M^*) \frac{\rho^+ - \mathrm{tr}M^*}{\mathrm{tr}\hat{M} - \mathrm{tr}M^*}\,.$$

   As $\frac{\rho^+ - \mathrm{tr}M^*}{\mathrm{tr}\hat{M} - \mathrm{tr}M^*} < 1$ it follows that $\tilde{M} \in \mathcal{M}$. Inserting into (50)

$$\begin{aligned}
&\int_{\tilde{\omega}} \langle \tilde{M} - M^*, \sum_{j=1}^{n} (-\lambda_j + \alpha^*) e_j {e_j}^\top \rangle \, dS = \\
&\quad = \int_{\tilde{\omega}} \frac{\rho^+ - \mathrm{tr}M^*}{\mathrm{tr}\hat{M} - \mathrm{tr}M^*} \langle \hat{M} - M^*, \sum_{j=1}^{n} (-\lambda_j + \alpha^*) e_j {e_j}^\top \rangle \, dS < 0
\end{aligned}$$

   yields a contradiction to the optimality of M^*. The same is true when assuming that there exists a $\hat{M} \in \tilde{\mathcal{C}} \setminus \mathcal{M}$ with $\mathrm{tr}\hat{M} < \rho^-$ such that

$$\int_{\tilde{\omega}} \langle \hat{M} - M^*, \sum_{j=1}^{n} (-\lambda_j + \alpha^*) e_j {e_j}^\top \rangle \, dS < 0\,.$$

   Then defining the matrix

$$\tilde{M} = M^* + (\hat{M} - M^*) \frac{\mathrm{tr}M^* - \rho^+}{\mathrm{tr}M^* - \mathrm{tr}\hat{M}} \qquad \in \mathcal{M}$$

   yields again a contradiction to the optimality of M^*:

$$\begin{aligned}
&\int_{\tilde{\omega}} \langle \tilde{M} - M^*, \sum_{j=1}^{n} (-\lambda_j + \alpha^*) e_j {e_j}^\top \rangle \, dS = \\
&\quad = \int_{\tilde{\omega}} \frac{\mathrm{tr}M^* - \rho^+}{\mathrm{tr}M^* - \mathrm{tr}\hat{M}} \langle \hat{M} - M^*, \sum_{j=1}^{n} (-\lambda_j + \alpha^*) e_j {e_j}^\top \rangle \, dS < 0\,.
\end{aligned}$$

   Thus we get the desired result

$$\int_{\tilde{\omega}} \langle M - M^*, \sum_{j=1}^{n} (-\lambda_j + \alpha^*) e_j {e_j}^\top \rangle \, dS \geq 0 \quad \forall\, M \in \tilde{\mathcal{C}}\,.$$

This completes the proof. □

Problem (26) is a convex nonlinear semidefinite program (SDP). Compared to the original problem formulation (22) problem (26) has several advantages. The matrices C and D are hidden in the problem as Lagrange multipliers. This significantly reduces the number of variables in the discrete problem (compare section 5). Furthermore problem (26) is convex. As t is the thickness of the shell, it is strictly positive and the matrix constraints of (26) can be simplified to

$$\begin{aligned} \frac{1}{2}\gamma(u)\gamma(u)^\top + \frac{t^2}{24}\chi(u,\theta)\chi(u,\theta)^\top - (\alpha+\beta_u-\beta_l)E_3 &\preceq 0\,, \\ \zeta(u,\theta)\zeta(u,\theta)^\top - (\alpha+\beta_u-\beta_l)E_2 &\preceq 0\,. \end{aligned} \tag{62}$$

5 Numerical Treatment

5.1 Discretization

We now intend to solve the infinite-dimensional SDP (26) numerically. For this purpose we discretize the problem by the finite element method [9]. The midsurface ω is partitioned into M elements ω_m. The number of corresponding element nodes is denoted by n. The elasticity matrices $C(x)$ and $D(x)$ are approximated by elementwise constant matrices $(C_1,\ldots,C_M)$ and $(D_1,\ldots,D_M)$ where $C_m \in \mathbb{R}^{3\times 3}$ and $D_m \in \mathbb{R}^{2\times 2}$ for all $m = 1,\ldots,M$. The displacements take the following form

$$U = \sum_{i=1}^{n} \lambda_i(r,s)\left(u^{(i)} + z\frac{t}{2}\theta^{(i)}\right)\,, \tag{63}$$

where the $\lambda_i(r,s)$ are bilinear 2D Lagrange shape functions. This assures that the Reissner-Mindlin assumption – material lines remain straight and unstretched during deformation – is fulfilled at all nodes of the mesh. Using (63) the discretized membrane strain matrix B_i^γ becomes

$$B_i^\gamma = \begin{pmatrix} \lambda_{i|1} & 0 & -b_{11}\lambda_i & 0 & 0 \\ 0 & \lambda_{i|2} & -b_{22}\lambda_i & 0 & 0 \\ \frac{1}{2}\lambda_{i|2} & \frac{1}{2}\lambda_{1|1} & -b_{12}\lambda_i & 0 & 0 \end{pmatrix}. \tag{64}$$

Therewith the discrete counterpart of the dyadic product $\gamma\gamma^\top$ reads

$$A_m^\gamma(u) = \sum_{i,j\in K}\int_{\omega_m} B_j^\gamma U U^\top (B_i^\gamma)^\top dx\,, \tag{65}$$

where K is the index set of nodes associated with the element m. Analogously we derive

$$A_m^\chi(u) = \sum_{i,j\in K}\int_{\omega_m} B_j^\chi U U^\top (B_i^\chi)^\top dx\,, \tag{66}$$

$$A_m^\zeta(u) = \sum_{i,j\in K}\int_{\omega_m} B_j^\zeta U U^\top (B_i^\zeta)^\top dx\,. \tag{67}$$

Replacing the forces and moments in problem (26) by their discrete counterparts we get the following discrete single load FMO problem for shells:

$$
\begin{aligned}
\max_{\substack{(u,\theta)\in\mathcal{U}\\ \alpha\in\mathbb{R}_0^+\\ \beta_u,\beta_l\in\mathbb{R}_0^{+\,M}}} \quad & \sum_{i=1}^{n}\left(tf_i u_i - \rho^+\beta_{ui}\right) + \sum_{i\in\partial\omega_1}(g_i u_i + h_i\theta_i)dl - V\alpha \\
\text{subject to} \quad & \frac{1}{2}A_m^\gamma(u) + \frac{t^2}{24}A_m^\chi(u,\theta) - (\alpha+\beta_u-\beta_l)E_3 \preceq 0 \\
& A_m^\zeta(u,\theta) - (\alpha+\beta_u-\beta_l)E_2 \preceq 0\,.
\end{aligned}
\tag{68}
$$

Obviously (68) is a finite-dimensional convex nonlinear semidefinite program.

5.2 Examples

Two numerical examples are presented in this section. In order to solve problem (68) we have used the nonlinear SDP code PENNON [15]. Although only the resulting "density" function $\mathrm{tr}(C) + \frac{1}{2}\mathrm{tr}(D)$ is depicted, we want to emphasize that the code provides the optimal elasticity matrices C_m and D_m for each element ω_m, $m = 1,\ldots,M$. Thus we gather information about the optimal material symmetry and material directions usable in the manufacturing process.

Example 1.

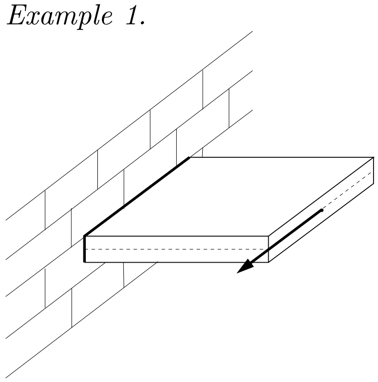

Fig. 2: Michell truss load case

The first example serves as a test for the consistency with the two-dimensional solid case. We consider a rectangular plate with in-plane forces. The plate is clamped on one side while forces are applied in the center of the opposite edge and directed in parallel to the boundary. This example known as Michell truss is widely used in topology optimization literature.

The typical material distribution of a Michell truss can be easily recognized in the displayed "density" distribution. It is also notable that only membrane strains appear as there is no deformation outside the midsurface. Thus there is no material used for the matrix D which accounts to shear effects.

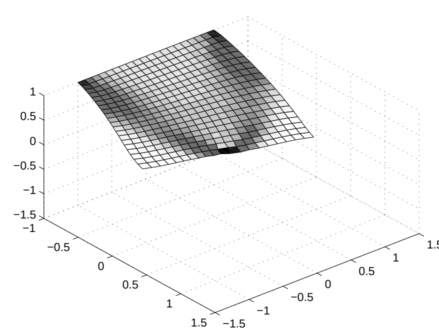

Fig. 3: Michell truss "density distribution"

Example 2.

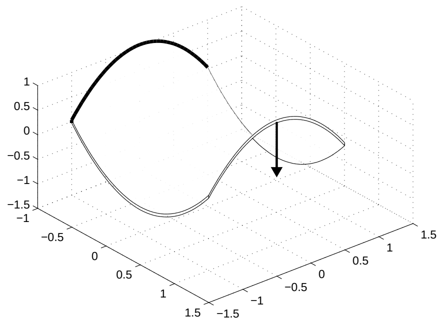

Fig. 4: Saddle-shaped hook load case

The second example employs all degrees of freedom of the shell. We start with a saddle-shaped midsurface, that is clamped on one side. A vertical force acts in the center of the opposite edge. This example can be interpreted as optimization of a coat hook fixed to the wall.

The resulting "density distribution" shows a firm tip at the location of the load. The shell tries to avoid vertical bending and distributes material over the complete design domain (apart from the corners in the front which are not suited to stiffen this particular structure). No holes can thus be found in contrast to the previous example. This result is not unexpected as stiff triangle-shaped structures appearing in the plane of loading are well known in topology optimization.

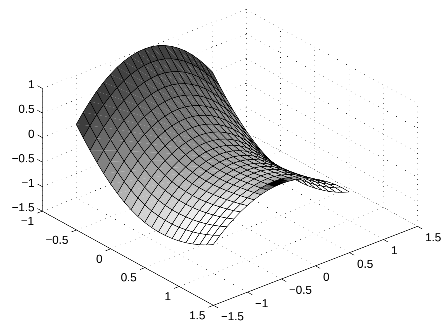

Fig. 5: Saddle-shaped hook "density distribution"

A Proof of Lemma 1 and 2

Proof of Lemma 1. If for reflexive two Banach spaces $\mathcal{V}$ and $\mathcal{Z}$ the following assumptions hold:

(I) $\mathcal{A} \subset \mathcal{V}$ and $\mathcal{B} \subset \mathcal{Z}$ are convex, closed and non-empty,

(II) the function $J : \mathcal{A} \times \mathcal{B} \to \mathbb{R}$ satisfies

$$\begin{aligned} &\forall\, p \in \mathcal{A} \qquad u \mapsto J(p,u) \text{ is concave and u.s.c.,} \\ &\forall\, u \in \mathcal{B} \qquad p \mapsto J(p,u) \text{ is convex and l.s.c.,} \end{aligned}$$

(III) and in addition

$$\begin{aligned} &\forall\, p \in \mathcal{A} \qquad u \mapsto J(p,u) \text{ is Gâteaux-differentiable,} \\ &\forall\, u \in \mathcal{B} \qquad p \mapsto J(p,u) \text{ is Gâteaux-differentiable,} \end{aligned}$$

then $(p^*, u^*) \in \mathcal{A} \times \mathcal{B}$ is a saddle-point of J according to [12, Chapter IV, Proposition 1.6] if and only if

$$\begin{aligned} \delta_p J(p^*, u^*)(p - p^*) &\geq 0 \quad \forall\, p \in \mathcal{A}\,, \\ \delta_u J(p^*, u^*)(u - u^*) &\leq 0 \quad \forall\, u \in \mathcal{B}\,. \end{aligned}$$

While in the present case $\mathcal{V} = [L^\infty(\omega)]^{3\times 3} \times [L^\infty(\omega)]^{2\times 2}$ is not a reflexive space this is irrelevant as the reflexivity of $\mathcal{V}$ is not used in the proof of [12, Chapter IV, Proposition 1.6]. In the case $\mathcal{A} = \mathcal{C}$, $\mathcal{B} = \mathcal{U}$ and $J = J_D$ the assumptions (I) and (II) are thus valid due to the proof of Theorem 1. For $\mathcal{A} = \tilde{\mathcal{C}}$, $\mathcal{B} = \mathcal{U} \times \mathcal{L}$ and $J = J_L$ assumption (I) is fulfilled as $\tilde{\mathcal{C}}$ – the cone of semidefinite matrices in $\mathcal{V}$ – is convex, closed and empty. This is also true for $\mathcal{L}$ and thus for $\mathcal{U} \times \mathcal{L}$. As the functional J_L is generated by adding linear functionals to J_D the properties required for (II) are preserved. (III) follows obviously from (34) to (37). $\square$

Proof of Lemma 2. This proof is an extension to matrices of the original proof in [7].

(i) First define a linear continuous mapping $\phi(M) : [L^\infty(\omega)]^{n\times n} \to \mathbb{R}^2$:

$$\phi(M) = \begin{pmatrix} \int_\omega {s_1}^\top M s_1 + {s_2}^\top M s_2 + \ldots + {s_p}^\top M s_p \, dS \\ \int_\omega \mathrm{tr} M \, dS \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}. \tag{69}$$

The range of ϕ is denoted by

$$\begin{aligned} G \; := \; \{(\xi_1, \xi_2) \in \mathbb{R}^2 \,|\, (\xi_1, \xi_2) = \phi(M)\,,\; M \in [L^\infty(\omega)]^{n\times n}\,, \\ M = M^\top \succeq 0\,,\; \rho^- \leq \mathrm{tr}(M) \leq \rho^+\}\,. \end{aligned} \tag{70}$$

The set G is closed, convex and bounded. The point $({\xi_1}^*, {\xi_2}^*)$ such that ${\xi_1}^*$ is maximal for ${\xi_2}^* = V$ corresponds to M^*. Thus $({\xi_1}^*, {\xi_2}^*) \in \partial G$ and there exists a hyperplane supporting G at $({\xi_1}^*, {\xi_2}^*)$ due to [23, Theorem 11.6]. Defining this hyperplane by the vector $(1, -\alpha^*)$ results in

$${\xi_1}^* - \alpha^* {\xi_2}^* \geq \xi_1 - \alpha^* \xi_2 \quad \forall\, \xi_1, \xi_2 \in G\,. \tag{71}$$

Inserting the definition of ξ_1 and ξ_2 yields

$$\int_\omega \sum_{i=1}^p {s_i}^\top M^* s_i \, dS - \alpha^* \int_\omega \mathrm{tr} M^* \, dS \geq \int_\omega \sum_{i=1}^p {s_i}^\top M s_i \, dS - \alpha^* \int_\omega \mathrm{tr} M \, dS \tag{72}$$

$$\forall\, M \in \mathcal{M} := \{M \in [L^\infty(\omega)]^{n\times n} \,|\, M = M^\top \succeq 0\,,\; \rho^- \leq \mathrm{tr}(M) \leq \rho^+\}.$$

Using the notation of the inner product this yields (50) and thus (i):

$$\int_\omega \langle M^*, \sum_{i=1}^p s_i {s_i}^\top - \alpha^* E_n \rangle \, dS \geq \int_\omega \langle M, \sum_{i=1}^p s_i {s_i}^\top - \alpha^* E_n \rangle \, dS \quad \forall\, M \in \mathcal{M}\,. \tag{73}$$

(ii) Suppose there exists a subset $\tilde{\omega} \subset \omega$, $|\tilde{\omega}| > 0$ where $\mathrm{tr}M^* > \rho^-$ a.e. We first show, that there exists at least one eigenvalue $\lambda_k > 0$, $k \in \{1, \dots, n\}$. Suppose $\lambda_j = 0 \, \forall j = 1, \dots, n$. Without loss of generality we can assume that there exists another subset $\hat{\omega} \subset \omega$, $|\hat{\omega}| > 0$, $\tilde{\omega} \cap \hat{\omega} = \emptyset$ with $\mathrm{tr}M^* < \rho^+$ a.e. in $\hat{\omega}$ and at least one $\lambda_k > 0$, $k \in \{1, \dots, n\}$ (which is true for non-vanishing s_i and a reasonable choice of ρ^+). Consider now the matrix

$$\hat{M} = M^* - \chi_{\tilde{\omega}} \frac{\mathrm{tr}M^* - \rho^-}{\mathrm{tr}M^*} M^* + \chi_{\hat{\omega}} \min \left\{ \frac{1}{|\hat{\omega}|} \int_{\tilde{\omega}} \mathrm{tr}M^* - \rho^- \, dS, \rho^+ - \mathrm{tr}M^* \right\} e_k {e_k}^\top ,$$

where $\chi_{\tilde{\omega}}(x)$ and $\chi_{\hat{\omega}}(x)$ are the characteric functions of the sets $\tilde{\omega}$ and $\hat{\omega}$, respectively. Thus it follows that $\hat{M} \in [L^\infty(\omega)]^{n \times n}$, $\hat{M} = \hat{M}^\top \succeq 0$, $\int_\omega \mathrm{tr}\hat{M} \, dS \leq V$, $\rho^- \leq \mathrm{tr}\hat{M} \leq \rho^+$ and we get

$$\begin{aligned}
\int_\omega \langle \hat{M}, \sum_{i=1}^p s_i {s_i}^\top \rangle dS \;&=\; \int_\omega \langle M^*, \sum_{i=1}^p s_i {s_i}^\top \rangle dS \\
&\quad - \int_{\tilde{\omega}} \frac{\mathrm{tr}M^* - \rho^-}{\mathrm{tr}M^*} \underbrace{\langle M^*, \sum_{j=1}^n \lambda_j e_j {e_j}^\top \rangle}_{=0} \, dS \\
&\quad + \int_{\hat{\omega}} \underbrace{\min \left\{ \frac{1}{|\hat{\omega}|} \int_{\tilde{\omega}} \mathrm{tr}M^* - \rho^- \, dS, \rho^+ - \mathrm{tr}M^* \right\}}_{>0} \cdot \underbrace{\lambda_k}_{>0} \, dS \\
&> \int_\omega \langle M^*, \sum_{i=1}^p s_i {s_i}^\top \rangle dS \, .
\end{aligned}$$

This obviously contradicts the optimality of M^*, thus for each $\tilde{\omega}$ with $\mathrm{tr}M^* > \rho^-$ there exists at least one positive eigenvalue. Assume now that for all $j \in \{1, \dots, n\}$ with $\lambda_j > 0$ $e_j \in \ker(M^*)$ holds. Consider another matrix

$$\tilde{M} = \begin{cases} M^* & \text{on } \omega \setminus \tilde{\omega} \\ \mathrm{tr}M^* e_k {e_k}^\top & \text{on } \tilde{\omega} \end{cases} .$$

In this case we get

$$\begin{aligned}
\int_\omega \langle \hat{M}, \sum_{i=1}^p s_i {s_i}^\top \rangle dS \;&=\; \int_{\omega \setminus \tilde{\omega}} \langle M^*, \sum_{i=1}^p s_i {s_i}^\top \rangle dS \\
&\quad + \underbrace{\int_{\tilde{\omega}} \langle M^*, \sum_{j=1}^n \lambda_j e_j {e_j}^\top \rangle dS}_{=0} + \int_{\tilde{\omega}} dS \, .
\end{aligned}$$

Again this contradicts the optimality of M^*. Thus for every $\tilde{\omega}$ with $\mathrm{tr}M^* > \rho^-$ a. e. there exists at least one $e_j \notin \ker(M^*)$ with a positive eigenvalue $\lambda_j > 0$. This shows (ii). Note that the optimality of M^* remains when neglecting all components belonging to the kernel of M^*:

$$\begin{aligned}
\int_\omega \langle M^*, \sum_{\substack{j=1\\ e_j\notin\ker(M^*)}}^n -\lambda_j e_j {e_j}^\top\rangle dS &= \int_\omega \langle M^*, \sum_{j=1}^n -\lambda_j e_j {e_j}^\top\rangle dS \\
= \quad \int_\omega \langle M^*, \sum_{i=1}^p -s_i {s_i}^\top\rangle dS &\le \int_\omega \langle M, \sum_{i=1}^p -s_i {s_i}^\top\rangle dS \\
= \quad -\int_\omega \langle M, \sum_{\substack{j=1\\ e_j\notin\ker(M^*)}}^n -\lambda_j e_j {e_j}^\top\rangle dS &- \underbrace{\int_\omega \langle M, \sum_{\substack{j=1\\ e_j\in\ker(M^*)}}^n -\lambda_j e_j {e_j}^\top\rangle\, dS}_{>0\ \forall M\in\mathcal{C}} \\
\le \quad -\int_\omega \langle M, \sum_{\substack{j=1\\ e_j\notin\ker(M^*)}}^n -\lambda_j e_j {e_j}^\top\rangle dS\,.
\end{aligned}$$

If all $e_j \notin \ker(M^*)$ have the same eigenvalue $\lambda > 0$ (this includes the case when there exists only one $e_j \notin \ker(M^*)$ with positive eigenvalue), then the problem can be projected onto the lower-dimensional space $\mathcal{P}$ spanned by these eigenvectors. Let $d_\mathcal{P}$ be the dimension of $\mathcal{P}$, then it follows that

$$\sum_{\substack{j=1\\ e_j\notin\ker(M^*)}}^n \lambda e_j {e_j}^\top = E_{d_\mathcal{P}} \quad \Rightarrow \quad \langle M, \sum_{\substack{j=1\\ e_j\notin\ker(M^*)}}^n -\lambda e_j {e_j}^\top\rangle = \lambda\, \mathrm{tr}_\mathcal{P} M\,. \tag{74}$$

Hence this case resembles the scalar case presented in [7]. Thus in the following we will assume that there exist at least two eigenfunctions $e_k, e_l \notin \ker(M^*)$ with different eigenvalues, one of them positive: $\lambda_k > \lambda_l \ge 0$.

(iii) Assume there exists a subset $\tilde\omega \subset \omega$, $|\tilde\omega| > 0$ with

$$\max_{\substack{j=1,\dots,n\\ e_j\notin\ker(M^*)}} \lambda_j < \alpha^* \quad \text{and} \quad \mathrm{tr}M^* > \rho^- \ \text{ a. e. in } \tilde\omega\,. \tag{75}$$

According to (ii) there exists at least one eigenfunction $e_k \notin \ker(M^*)$ with $\lambda_k > 0$, $k \in \{1,\dots,n\}$. Furthermore we assume, there exists at least one other eigenfunction $e_l \notin \ker(M^*)$ and $\lambda_k > \lambda_l \ge 0$, $l \in \{1,\dots,n\}$, otherwise this would represent the scalar case from [7]. Define the matrix $\tilde M$ by

$$\tilde M = M^* + \chi_{\tilde\omega}(x)\min\{\mathrm{tr}M^* - \rho^-, {e_l}^\top M^* e_l\}\left(e_k {e_k}^\top - e_l {e_l}^\top\right)\,.$$

This yields not only $\operatorname{tr}\tilde{M} = \operatorname{tr}M^*$, but also $\tilde{M} \in \mathcal{M}$. Consider now

$$\begin{aligned}
&\int_\omega \langle \tilde{M}, \sum_{i=1}^p s_i {s_i}^\top - \alpha^* E_n \rangle dS \\
= &\int_\omega \langle M^*, \sum_{i=1}^p s_i {s_i}^\top \rangle - \alpha^* \operatorname{tr}\tilde{M}\, dS \\
&+ \int_\omega \chi_{\tilde{\omega}} \min\{\operatorname{tr}M^* - \rho^-, {e_l}^\top M^* e_l\} \langle e_k {e_k}^\top - e_l {e_l}^\top, \sum_{j=1}^n \lambda_j e_j {e_j}^\top \rangle dS \\
= &\int_\omega \langle M^*, \sum_{i=1}^p s_i {s_i}^\top \rangle - \alpha^* \operatorname{tr}M^*\, dS + \int_{\tilde{\omega}} \underbrace{\min\{\operatorname{tr}M^* - \rho^-, {e_l}^\top M^* e_l\}}_{>0} \underbrace{(\lambda_k - \lambda_l)}_{>0} dS \\
> &\int_\omega \langle M^*, \sum_{i=1}^p s_i {s_i}^\top - \alpha^* E_n \rangle dS\,.
\end{aligned}$$

As this is a contradiction to the optimality of M^*, this yields (iii).

(iv) The proof for the case $\min_{\substack{j=1,\dots,n \\ e_j \notin \ker(M^*)}} \lambda_j > \alpha^*$ works analogously with the matrix

$$\tilde{M} = M^* + \chi_{\tilde{\omega}}(x) \min\{\rho^+ - \operatorname{tr}M^*, {e_l}^\top M^* e_l\} \left(e_k {e_k}^\top - e_l {e_l}^\top \right) .$$

Although the case $\operatorname{tr}M^* < \rho^+$ includes $\operatorname{tr}M^* = \rho^-$, no problems occur, as either there is a contradiction to $\min_{\substack{j=1,\dots,n \\ e_j \notin \ker(M^*)}} \lambda_j > \alpha^*$ or there exists an eigenvalue $\lambda_k > 0$ with an associated eigenfunction $e_k \notin \ker(M^*)$ and the proof of (iv) again holds.

$\square$

References

[1] A. Ben-Tal, M. Kočvara, A. Nemirovski, and J. Zowe. Free material design via semidefinite programming: The multiload case with contact conditions. *Siam J. Optim.*, 9(4):813–832, 1999.

[2] M. P. Bendsœ and A. R. Díaz. Optimization of material properties for mindlin plate design. *Structural Optimization*, 6:268–270, 1993.

[3] M. P. Bendsœ, J. M. Guedes, R. B. Haber, P. Pedersen, and J. E. Taylor. An analytical model to predict optimal material properties in the context of optimal structural design. *J. Appl. Mech. Trans. ASME*, 61:930–937, 1994.

[4] M. P. Bendsœ and O. Sigmund. *Topology Optimization. Theory, Methods and Applications.* Springer Verlag, Heidelberg, 2002.

[5] M. Blouza and H. Le Dret. Naghdi's shell model: Existence, uniqueness and continuous dependence on the midsurface. *Journal of Elasticity*, 64:199–216, 2001.

[6] N. Camprubí, M. Bischoff, and K.-U. Bletzinger. Shape optimization of shells and locking. *Computers and Structures*, 82:2551–2561, 2004.

[7] J. Cea and K. Malanowski. An example of a max-min problem in partial differential equations. *SIAM J. Control*, 8(3):305–316, 1970.

[8] P. Cervellera, M. Zhou, and U. Schramm. Optimization driven design of shell structures under stiffness, strength and stability requirements. In *6th World Congresses of Structural and Multidisciplinary Optimization*, Rio de Janeiro, Brazil, 2005.

[9] D. Chapelle and K. J. Bathe. *The Finite Element Analysis of Shells – Fundamentals*. Springer, Heidelberg, 2003.

[10] P. G. Ciarlet. *Mathematical Elasticity. Volume 3: Theory of Shells*. North Holland, Amsterdam, 2000.

[11] E. Cosserat and F. Cosserat. *Théorie des Corps Déformables*. Hermann, Paris, 1909.

[12] I. Ekeland and R. Temam. *Convex Analysis and Variational Problems*. North Holland, Amsterdam, 1976.

[13] I. Figueiredo and C. Leal. Ellipticity of Koiter's and Naghdi's models for nonhomogeneous anisotropic shells. *Applicable Analysis*, 70:75–84, 1998.

[14] H. R. E. M. Hörnlein, M. Kočvara, and R. Werner. Material optimization: Bridging the gap between conceptual and preliminary design. *Aerospace Science and Technology*, 5(8):541–554, 2001.

[15] M. Kočvara and M. Stingl. PENNON - a code for convex nonlinear and semidefinite programming. *Optimization Methods and Software*, 18(3):317–333, 2003.

[16] M. Kočvara and M. Stingl. Solving nonconvex SDP problems of structural optimization with stability control. *Optimization Methods and Software*, 19(5):595–609, 2004.

[17] M. Kočvara and M. Stingl. Free material optimization: Towards the stress constraints. *Structural and Multidisciplinary Optimization*, 2006. DOI10.1007/s00158-007-0095-5.

[18] T. Lewiński and J. Sokołowski. Optimal shells formed on a sphere. the topological derivative method. Technical Report RR-3495, INRIA-Lorraine, 1998.

[19] A. Lipka. *Verbesserter Materialeinsatz innovativer Werkstoffe durch die Topologieoptimierung*. PhD thesis, Universität Stuttgart, Institut für Baustatik und Baudynamik, 2007.

[20] J. Mach. Finite element analysis of free material optimization problem. *Applications of Mathematics*, 49(4):285–307, 2004.

[21] K. Maute and E. Ramm. Adaptive topology optimization of shell structures. *AIAA*, 35(11):1767–1773, 1997.

[22] P. M. Naghdi. The theory of shells and plates. In *Handbuch der Physik VIa/2*, pages 425–640. Springer, 1972.

[23] R. T. Rockafellar. *Convex Analysis*. Princeton University Press, 1970.

[24] M. B. Rubin. *Cosserat Theories: Shells, Rods and Points*. Kluwer Academic Publishers, Dordrecht, 2000.

[25] J. Stegmann and E. Lund. Discrete material optimization of general composite shell structures. *Int. J. Numer. Meth. Engng*, 62:2009–2027, 2005.

[26] R. Werner. *Free Material Optimization - Mathematical Analysis and Numerical Solution*. PhD thesis, Friedrich-Alexander-Universität Erlangen-Nürnberg, Institut für Angewandte Mathematik II, 2001.

[27] J. Zowe, M. Kočvara, and M. P. Bendsœ. Free material optimization via mathematical programming. *Mathematical Programming Series B*, 79:445–466, 1997.